MIP Modeling of Incremental Connected Facility Location

Ashwin Arulselvan[1] Andreas Bley[2] Ivana Ljubić[3]

[1] University of Strathclyde, UK
[2] University of Kassel, Germany
[3] ESSEC Business School of Paris, France

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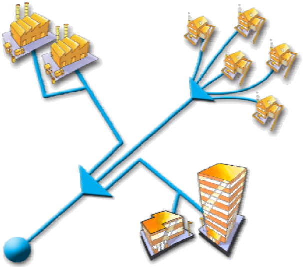
Fiber optic access networks



Next generation access networks

- Replace long copper lines by optical fibers to the curb/building (FTTC / FFTB / FTTA)
  - more bandwidth
  - more reach
  - less active locations
  - less energy consumption
- Nation-wide technology change
  - replacement of link technology on last miles to customers
  - huge investments
  - work intensive
  - deployment in multiple phases

Fiber optic access networks

Trail network with existing and potential infrastructure

Strategic planning

- Which facilities (COs) to open?
- When to migrate which region?
- Use of intermediate technologies and technology mix?

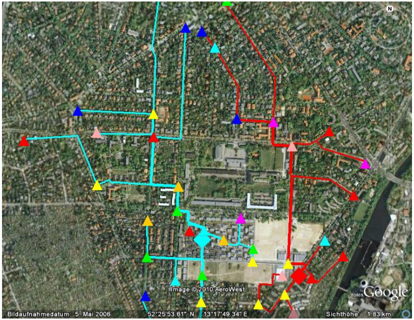
Company objective

- Maximize net present value

Main constraints

- network technically feasible
- meet minimum service levels (given by regulation authorities)

Fiber optic access networks



Part of an FTTH network

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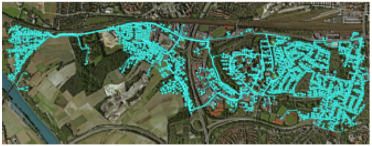
Fiber optic access networks

Year 1

Year 2



Year 3

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Deployment of new Access Networks

Typical real-world setting

- Mix of different (intermediate) technologies
  - ADSL, VDSL (copper lines), LTE (ratio), FTTB (optical fiber)
- Different service levels with coverage constraints

Year	available bandwidth		
	$\geq 384kb/s$	$\geq 7.2Mb/s$	$\geq 50Mb/s$
X	$\geq 80\%$	$\geq 10\%$	$\geq 0\%$
X+1	$\geq 90\%$	$\geq 30\%$	$\geq 10\%$
X+2	$\geq 95\%$	$\geq 50\%$	$\geq 25\%$
⋮		⋮	

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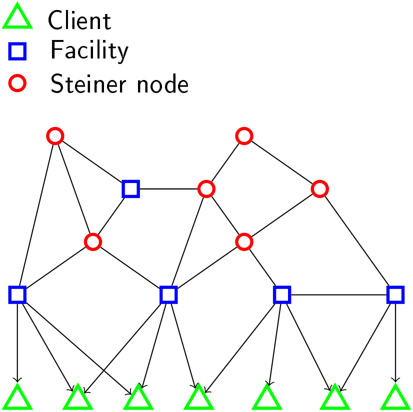
$\vdots$		$\vdots$	

Simplified model

- Only single 'new' technology
- Only direct client-facility connections
- Applicable to DSL → FTTx migration
  - global migration planning for many regions
  - fiber optic rollout within a small region

Incremental Connected Facility Location Problem



Given

- Mixed graph $G = (V, E)$ with

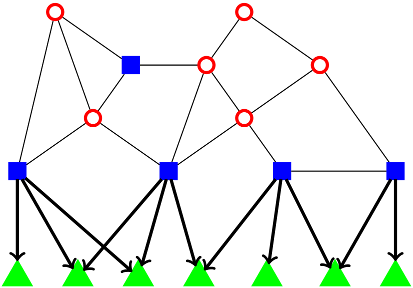
$$V = \begin{array}{l} R \text{ potential clients} \\ \left.\begin{array}{l} F \text{ potential facilities} \\ M \text{ Steiner nodes} \end{array}\right\} = S \end{array}$$

$$E = \begin{array}{l} A_R \text{ fac.-client connections} \\ E_S \text{ edges of facility network} \end{array}$$

- Planning horizon $T = 1, \dots, \mathcal{T}$
- Setup and maintenance costs for facilities and edges
- Demands and profits for clients
- Total demand D^t to be covered
- Discount factor for costs/profits

Incremental Connected Facility Location Problem

△ Client
□ Facility
○ Steiner node



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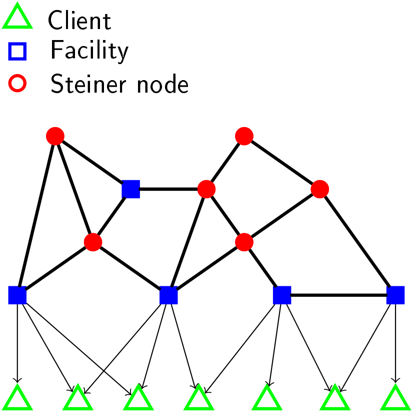

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  F potential facilities

  M Steiner nodes $\Big\} = S$

  $E = A_R$ fac.-client connections

  E_S edges of facility network

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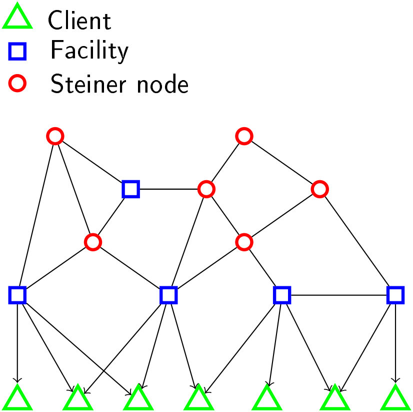

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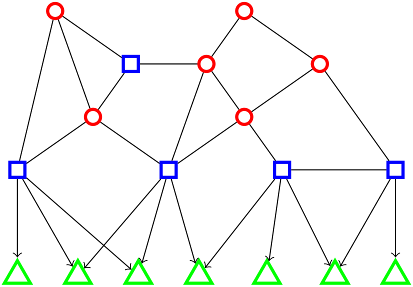

Seek

- For each time period $t \in T$:
  - facilities to open
  - clients to serve
  - edges connecting all open facilities and served clients

Such that

- total demand of clients served in period t exceeds D^t
- each client is served undisrupted (once served, always served)
- net present value is maximized

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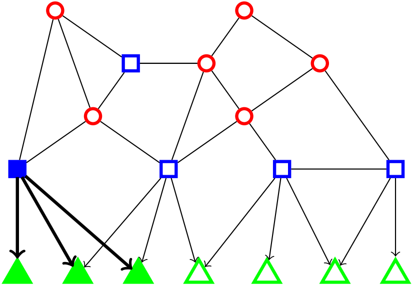

Period $t = 1$

Seek

- For each time period $t \in T$:
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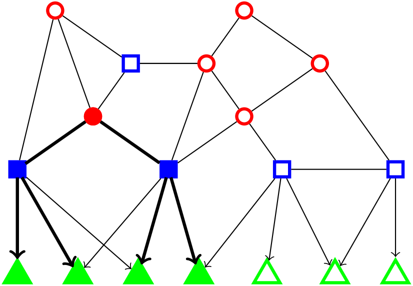

Period $t = 2$

Seek

- For each time period $t \in T$:
    - facilities to open
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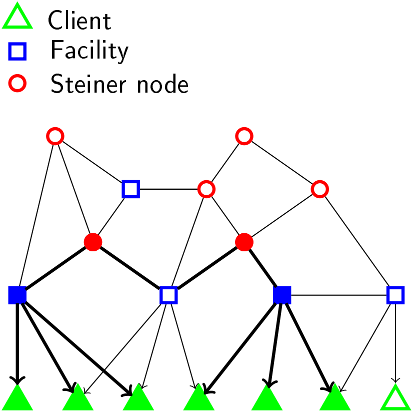
Incremental Connected Facility Location Problem



Period $t = 3$

Seek

- For each time period $t \in T$:
    - facilities to open
    - clients to serve
    - edges connecting all open facilities and served clients

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Incremental Connected Facility Location Problem

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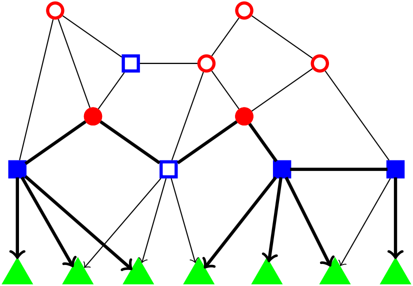

Period $t = 4$

Seek

- For each time period $t \in T$:
    - facilities to open
    - clients to serve
    - edges connecting all open facilities and served clients

Such that

- total demand of clients served in period t exceeds D^t
- each client is served undisrupted (once served, always served)
- net present value is maximized

Literature

- ConFL exact: Gollowitzer and Ljubić (2011); Leitner et al. (2017)
- ConFL heuristics: Eisenbrand et al. (2010); Bardossy and Raghavan (2010)
- incremental FL: Albareda-Sambola et al. (2009), Arulselvan et al. (2015)
- Incremental network design (shortest paths, spanning trees and maximum flows): Baxter et al. (2014); Engel et al. (2017); Kalinowski et al. (2015),

BASIC MILP FORMULATION

IP Modeling – Variables

Time-indexed ConFL model with

Annual maintenance cost for used facilities and used arcs

Annual revenues for served customers

Setup cost for used facilities and used edges

IP Modeling – Variables

Time-indexed ConFL model with binary variables for

Annual maintenance cost for used facilities and used arcs

- z_i^t: 1 if facility i is used in time period t, 0 otherwise
- x_{ij}^t: 1 if arc (i, j) is used in time period t, 0 otherwise

Annual revenues for served customers

- y_j^t: 1 if customer j is served in time period t, 0 otherwise

Setup cost for used facilities and used edges

IP Modeling – Variables

Time-indexed ConFL model with binary variables for

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Annual revenues for served customers

- y_j^t: 1 if customer j is served in time period t, 0 otherwise

Setup cost for used facilities and used edges

- $\tilde{x}_e^t(\tilde{z}_e^t)$: 1 if edge e (facility i) is used in time period t, 0 otherwise

IP Modeling – Objective

Maximize the net present value

$$\max \sum_{t=1}^{T} (1+\alpha)^{-t} \Bigg[\sum_{j \in R} p_j y_j^t \qquad \text{revenues}$$

$$- \sum_{i \in F} g_i \tilde{z}_i^t - \sum_{e \in E} c_e \tilde{x}_e^t \qquad \text{setup}$$

$$- \sum_{i \in F} m_i z_i^t - \sum_{(i,j) \in A} m_{ij} x_{ij}^t \Bigg] \qquad \text{maintenance}$$

IP Modeling – Objective

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Remark

- Here: discounted fixed revenues and costs
- Analogously: time-dependent revenue and cost functions

IP Modeling – Constraints

Standard ConFL constraints for each period $t \in T$

- Assign served clients

$$\sum_{i \in F(j)} x_{ij}^t = y_j^t \qquad \forall j \in R,\ t \in T \tag{1}$$

- Open assigned facilities

$$x_{ij}^t \leq z_i^t \qquad \forall (i,j) \in A_R,\ t \in T \tag{2}$$

- Connect root to open facilities

$$\sum_{(u,v) \in \delta^-(W)} x_{uv}^t \geq z_j^t \quad \forall W \subseteq S \backslash \{r\},\ j \in W \cap F,\ t \in T \tag{3}$$

IP Modeling – Constraints

Setup used edges and facilities

$$x_{ij}^t + x_{ji}^t \leq \sum_{k=1}^{t} \tilde{x}_e^k \qquad \forall (i,j) = e \in E,\ t \in T \qquad (4)$$

$$z_i^t \leq \sum_{k=1}^{t} \tilde{z}_i^k \qquad \forall i \in F,\ t \in T \qquad (5)$$

Minimum demand coverage

$$\sum_{j \in R} d_j y_j^t \geq D^t \qquad \forall t \in T \qquad (6)$$

No service withdrawal

$$y_j^t \geq y_j^{t-1} \qquad \forall j \in R,\ t \in T \qquad (7)$$

STRENGTHENING INEQUALITIES

Knapsack Cover Inequalities on Customers

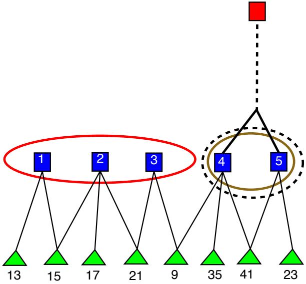
- Derived from the coverage constraint $\sum_{j \in R} d_j y_j \geq D$.
- Let J be the minimal subset of customers such that its complement $\bar{J} = R \setminus J$ cannot satisfy the whole demand, i.e., $D(\bar{J}) < D$ and $d_k + D(\bar{J}) \geq D$, for any $k \in J$.
- J is called the *minimal cover* wrt D. Let COV_R be the collection of all such minimal covers.
- *Knapsack cover inequalities* are valid for $\mathrm{P}(D)$:

$$y(J) \geq 1 \qquad J \in COV_R \tag{8}$$

Set-Union Knapsack Cover (SUKC) Inequalities

Consider facility location for single period t

- Example: $F \setminus \{r\} = \{1, \ldots, 5\}$, $D = 174$, $D^t = 76$.
- Valid LP solution: $z_i^t = 76/174$ for $i = 1, \ldots, 5$ (optimal for high core connection costs)



Demand requirement $D^t = 76$

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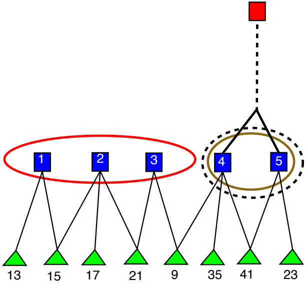
- Valid LP solution: $z_i^t = 76/174$ for $i = 1, \ldots, 5$ (optimal for high core connection costs)
- Combined demand of clients of $\{1, 2, 3\}$ is $75 < D^t$

$\Rightarrow$ Either 4 or 5 has to be open



Demand requirement $D^t = 76$

Definition

- $I^t \subset F$ is cover if $F \setminus I^t$ cannot serve enough clients to meet D^t.
- I^t is minimal cover if no cover $J^t \subsetneq I^t$ exists.
- $COV^t :=$ family of minimal covers for period t

SUKC Inequalities

Theorem

1. *The SUKC inequalities*

$$\sum_{i \in I^t} z_i^t \geq 1 \quad \forall t \in T,\ I^t \in COV^t \tag{9}$$

   are valid for the integer solutions of (1)–(7).

2. *There are problem instances and corresponding (optimal) fractional solutions for (1)–(7) that do not satisfy (9).*

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Separation

- Search for the complement of I^t which cannot cover all the demand
- Set union knapsack problem: Each item has a profit, and covers a set of elements. Each element has a weight.
- Find a subset of items of maximum profit that covers elements of weight at most B.

Separation of SUKC inequalities

Separation of SUKC inequalities

- Set union knapsack problem, strongly NP-hard Goldschmidt et al. (1994)
- Greedy gives a constant factor approximation for special case

$$\min \sum_{i \in F} \hat{z}_i^t \alpha_i$$

$$\sum_{j \in R} d_j \beta_j \leq D^t - \epsilon$$

$$\beta_j \geq 1 - \alpha_i \qquad \forall (i,j) \in A_R$$

$$\alpha_i, \beta_j \in \{0,1\} \qquad \forall i \in F, j \in R$$

  - $\hat{z}^t$ – current (fractional) values of facility variables
  - $\alpha_i \in \{0,1\}, \forall i \in F$: $\alpha_i = 1$ iff facility in I^t
  - $\beta_j \in \{0,1\}, \forall j \in R$: $\beta_j = 1$ iff client j can be served by $F \setminus I^t$

Separation of SUKC inequalities

Heuristic investigations of the separation problem

```
$H := \emptyset$
Let $w_j$ be the number of facilities serving customer $j$, $j \in R$.
while $F \neq \emptyset$ do
    pick a facility $i \in F$ that maximizes $\frac{\hat{z}_i}{\sum_{j \in R_i} \frac{d_j}{w_j}}$ and remove it from $F$
    if $D(R_H) + D(R_i) < D$ then
        $H := H \cup \{i\}, \quad R := R \setminus \{R_i\}$
return SUKC $I := F \setminus H$
```

- Greedy heuristic provides approximation guarantees for special cases Arulselvan (2014)
  - Number of facilities serving a customer is bounded
- Cover complements of size 2 and 3 are enumerated and added

Cut-set-SUKC Inequalities

- Suppose $z_i^t = 1/|I^t|$ for all facilities i in some cover I^t.
- Then LP enforces only connectivity $1/|I^t|$ from r to I^t.
- But: I^t must be 1-connected from r in ILP solution.

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Theorem

1. *The cut-set cover inequalities*

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are valid for the integer solutions of (1)–(7).

2. *There are problem instances and corresponding (optimal) fractional solutions for* (1)–(7), (9) *that do not satisfy* (10).

SOME LIFTING IDEAS

Lifting

- We try to lift SUKC inequalities, by including customer variables.
- Denote by $R_2 \subseteq R$ the subset of customers j such that if j is not served, we need to open at least two facilities from I in order to serve the demand D.

$$z(I) \geq 2 - y_j \quad I \in COV_F, j \in R_2 \tag{11}$$

- In general

$$z(I) + \sum_{k \in L} \alpha_k (y_k - 1) + \alpha_j (y_j - 1) \geq 1 \tag{12}$$

COMPUTATIONAL RESULTS

Implementation

- Branch-and-cut based on CPLEX 12.2 with Python API
- Initial formulation
    - all model constraints except connectivity cuts
    - indegree inequalities
- Separation via cut callback
    - root—facility cut-set inequalities
    - SUKC and customer cover inequalities
    - cut-set-SUKC inequalities
    - lifted inequalities

Implementation

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  - lifted inequalities
- Branching priorities
  - facility vars z_i^t > client vars y_j^t > remaining (arc) vars

Primal heuristics

Let (x^*, y^*, z^*) be the optimal LP value at a branch and bound node

- Round up the fractional y^* and z^* variables for all time periods
- Assign the customers to its cheapest fractional facility
- Post-processing to remove non-serving open facilities
- Facility open at time t remains open for subsequent time period
- Shortest path heuristic for Steiner tree problem for period T
  - Take the cost of edge (i, j) to be $(1 - x^*_{ij})$
  - Consider the facilities in some order
  - Find the shorted path from a terminal to the root node
  - Fix the cost of the edges in this path to be zero
- For time periods, $T - 1, T - 2, \ldots,$ remove the paths of missing facilities

Test instances

- Benchmark instances
  - Combination of UFL and Steiner tree instances
  - UFL: mp1, mp2, mp3 (200x200, 300x300)
  - STP: 500-1,000 nodes, up to 25,000 edges
  - Sparsification: only 20 closest facilities per client

  $\Rightarrow$ up to 1,300 nodes and 45,000 edges
- time horizon $\mathcal{T} = 3$
- client demands random in [20-40]
- min coverage $\{50\%, 75\%, 90\%\}$
- 8 real world instances of varying sizes

	B				B^c			
	root gap	Final Gap	Cuts	nodes	root gap	Final Gap	Cuts	nodes
a	77.6	32.7	17409	313	81.0	77.2	9275/1075	1111
b	71.3	67.5	12128	488	77.3	58.0	5937/220	240
c	67.1	59.4	12976	1651	68.8	60.4	8121/1076	1123
d	57.4	57.4	8982	0	63.5	63.5	3710/69	20
e	63.9	63.9	10171	5	64.9	64.9	4894/90	47
f	64.9	64.9	12756	20	67.7	67.7	5455/100	52
g	57.2	57.2	7353	0	63.2	63.2	3703/69	20
h	78.9	78.6	1432	0	78.1	78.1	1939/36	0

	B^+				B^{++}			
	root gap	Final Gap	Cuts	nodes	root gap	Final Gap	Cuts	nodes
a	4.0	1.6	6528/411/3410	385	4.0	2.4	4265/184/1965/6370	145
b	2.4	1.2	1844/411/851	400	2.4	1.2	1644/267/854/2849	250
c	4.0	0.2	5183/153/1654	98	4.4	0.3	3055/410/2280/13058	370
d	23.1	23.1	6199/32/442	0	25.9	25.9	1611/30/476/974	0
e	12.4	12.4	7513/39/587	2	12.5	12.5	2561/47/1133/0	0
f	37.5	37.5	10403/53/1454	5	40.6	40.6	2744/50/1105/417	0
g	24.0	24.0	6140/32/557	0	26.1	26.1	1544/29/458/1059	0
h	63.8	63.8	3675/18/111	0	69.5	69.5	898/16/88/647	0

Comparison of four branch-and-cut settings for the ConFL-CR with 50% coverage.

	B				B^c			
	root gap	Final Gap	Cuts	nodes	root gap	Final Gap	Cuts	nodes
a	70.4	10.4	18857	220	74.7	71.2	10573/332	326
b	64.4	61.8	13159	396	63.6	60.1	6750/387	354
c	62.0	55.7	15523	1071	63.4	55.7	8415/1358	1352
d	47.8	47.8	8066	0	53.6	53.6	3519/65	15
e	53.8	53.8	10392	5	55.9	55.4	3973/74	30
f	56.7	56.7	12209	13	59.9	59.9	5887/109	60
g	49.1	49.1	8062	0	53.9	53.9	3700/69	20
h	67.6	67.6	5285	0	72.1	72.0	2176/41	0

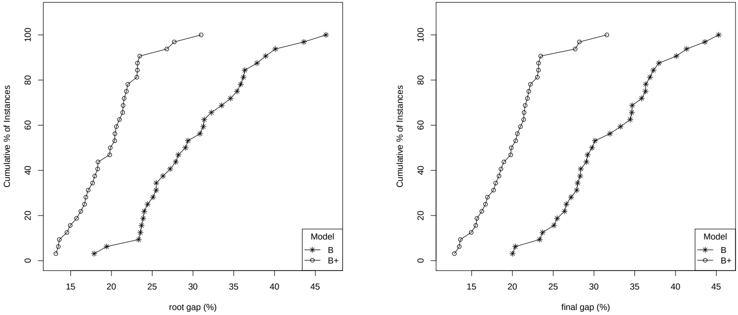
	B^+				B^{++}			
	root gap	Final Gap	Cuts	nodes	root gap	Final Gap	Cuts	nodes
a	7.9	5.2	11148/221/3030	195	7.9	6.9	6363/203/3228/1625	173
b	5.7	5.6	2622/520/1057	496	5.5	5.5	1588/502/694/3216	465
c	2.3	0.0	2334/94/419	68	2.1	0.3	2374/96/420/101	61
d	22.5	22.5	6646/33/336	0	24.5	24.5	2009/37/762/379	0
e	13.4	13.4	8863/56/811	10	13.6	13.6	3049/58/1568/246	12
f	35.7	35.7	12128/60/1766	10	37.9	37.9	3508/65/2021/1705	17
g	21.8	21.7	5788/29/465	0	25.0	25.0	1860/35/688/440	0
h	60.2	60.2	4300/21/246	0	66.7	66.7	1262/24/266/1322	0

Comparison of four branch-and-cut settings for the ConFL-CR with 75% coverage.

	B				B^c			
	root gap	Final Gap	Cuts	nodes	root gap	Final Gap	Cuts	nodes
a	67.6	59.4	21032	127	69.9	66.6	9776/806	767
b	59.3	56.4	14730	240	62.7	56.9	5167/496	478
c	56.5	51.0	15631	1046	58.0	52.1	7936/1016	1008
d	47.3	47.3	6734	0	51.0	51.0	2692/50	0
e	46.6	46.6	10346	5	49.8	49.8	4300/78	30
f	49.6	49.6	12246	11	52.7	52.7	5108/94	45
g	48.7	48.7	7356	0	51.9	51.9	2664/50	0
h	61.1	61.1	5304	0	65.6	65.6	2373/44	0

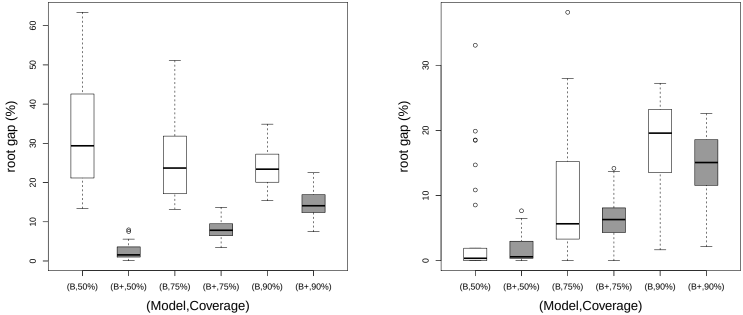
	B^+				B^{++}			
	root gap	Final Gap	Cuts	nodes	root gap	Final Gap	Cuts	nodes
a	7.7	5.6	14597/174/9666	142	10.2	7.2	6510/177/10285/685	150
b	2.6	2.4	2607/508/685	286	2.6	2.5	2813/532/578/8570	500
c	4.0	0.4	4237/115/1225	94	4.0	0.0	2711/87/773/5	68
d	25.5	25.5	5704/28/588	0	28.2	28.2	2057/38/1030/7	0
e	10.0	10.0	8378/46/543	8	9.8	9.8	3055/58/1238/164	15
f	28.2	28.2	10654/53/1334	7	28.8	28.8	3488/65/2010/565	15
g	27.2	27.2	5112/25/397	0	29.5	29.5	1882/35/879/7	0
h	55.0	55.0	4079/20/307	0	61.0	61.0	1376/26/484/989	0

Comparison of four branch-and-cut settings for the ConFL-CR with 90% coverage.



(a) (b)

Three-period iConFL results for coverage rates: 50%, 75% and 90%. ConFL instances.

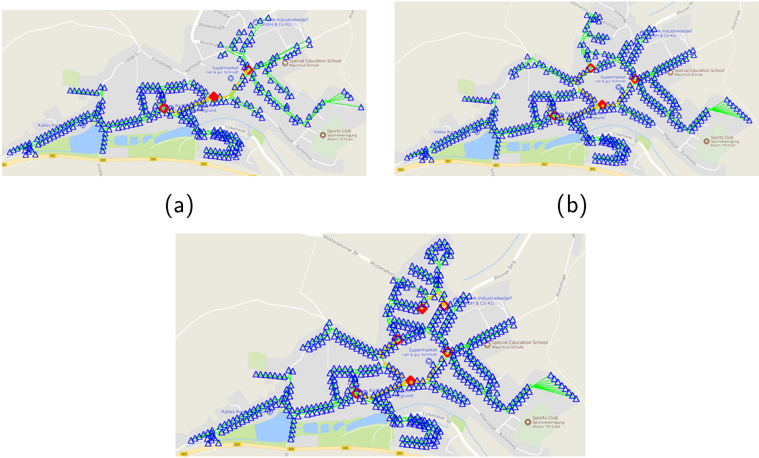


(a) Gap at the root node.

(b) Final gap.

Comparison of root- and final gaps with increasing coverage rates. ConFL instances.

Experiments



(a)

(b)

(c)

Three-period iConFL results for coverage rates for Ahorn: 50%, 75% and 90%. ConFL instances.

Conclusions

Summary

- Incremental connected facility location
- Valid inequalities and strengthening
- Experiments and results

Interesting questions

- Approximation algorithm for budgeted version of connected facility location
- Buy-at-bulk network design problems in incremental and budgeted settings

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