Generalized Benders Cuts for Congested Facility Location

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Capacitated facility location (CFL) with multiple allocation

- Given:
  - bipartite graph $G = (I \cup J, E)$,
  - J: potential facility locations, I: customers, E possible allocations
  - Customers to be served by open facilities.
  - Demand $d_i > 0$ for each customer $i \in I$.
  - Capacity $s_j > 0$, for each facility $j \in J$.
  - Demand can be split and a customer partially served by several facilities.
  - Facility opening costs $f_j > 0$, allocation costs $c_{ij} > 0$ (per unit of demand)
- Goal: find facilities to open and allocate customers to minimize

  costs for opening facilities plus allocation costs.
- NP-hard problem (uncapacitated problem: reduction from set-cover)

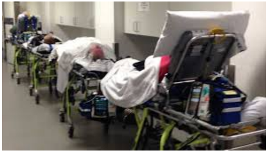
CFL: MIP-model

- binary variables $y_i = 1$, iff facility i is opened
- variables x_{ij} fraction of demand of customer i served by facility j

$$
\begin{aligned}
\min \sum_{j\in J} f_j y_j + \sum_{i\in I}\sum_{j\in J} d_i c_{ij} x_{ij} \\
\text{s.t.} \quad \sum_{j\in J} x_{ij} &= 1 && \forall i \in I \\
0 \le x_{ij} &\le y_j && \forall i \in I, j \in J \\
\sum_{i\in I} d_i x_{ij} &\le s_j y_j && \forall j \in J \\
y_j &\in \{0,1\} && \forall j \in J
\end{aligned}
$$

- total customer demand is satisfied
- (partial) allocation to a facility j is only possible if this facility is open

Congested Capacitated Facility Location


- Congestion at facilities can lead to
  - huge delays,
  - higher cost: overtime workers, costly materials,
  - postponing/neglecting maintenance schedules.
- Congestion costs: diseconomies of scale!
- For example, in production-distribution networks, convex "costs":
  - service/production costs at facilities,
  - waiting times (not always measured in currencies)
  - number of waiting items

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 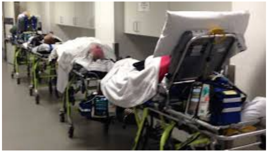 

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  - number of waiting items

How to model congestion costs?

- Queuing theory...
- As a convex function of the facility load: total demand served by a facility
- Let the load of facility j be:

$$v_j = \sum_{i \in I} d_i x_{ij}$$

- Congestion can be measured (see Desrochers et al., 1995)

$$v_j \cdot F(v_j)$$

  where F is a convex penalty function associated with the load.
- Desrochers et al., 1995: If F is non-negative, convex, increasing, so is

$$\sum_{i \in I} d_i x_{ij} \cdot F(\sum_{i \in I} d_i x_{ij}).$$

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Congested CFL: MINLP

$$\min \sum_{j\in J} f_j y_j + \sum_{i\in I}\sum_{j\in J} d_i c_{ij} x_{ij} + \sum_{j\in J} v_j \cdot F(v_j)$$

$$\text{s.t.} \quad \sum_{j\in J} x_{ij} = 1 \qquad \forall i \in I$$

$$0 \le x_{ij} \le y_j \qquad \forall i \in I, j \in J$$

$$\sum_{i\in I} d_i x_{ij} = v_j \qquad \forall j \in J$$

$$v_j \le s_j y_j \qquad \forall j \in J$$

$$\sum_{j\in J} y_j = p$$

$$y_j \in \{0,1\} \qquad \forall j \in J$$

p-median constraint: to avoid opening too many facilities

For a fixed value of y^*, the NLP is a convex problem. However, this continuous relaxation is not particularly strong.

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Congested CFL: Previous Work

- Introduced by Desrochers, Marcotte, Stan (Location Science, 1995): branch-and-price (pricing problem is a convex NLP).
- Instances of size 57×57, with $p = 13, 29, 55$
- ...
- Selfun, 2011: master thesis, Bilkent Univ. Outer approximation.
- Instances of size $|I| = |J| \in \{20, 40, 60, 80\}$ solved within 10 minutes

Our contribution:

1. Derive a **tighter MINLP** formulation (perspective reformulation)
2. Reduce the MINLP-size: remove x_{ij} and v_j variables (**generalized Benders decomposition**)
3. Solve the newly obtained **MILP using a branch-and-cut** (separate generalized Benders cuts on the fly)
4. Instances of size $|I| \times |J| \in \{300 \times 300, \dots, 1000 \times 1000\}$ - solved to optimality.

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STEP 1: PERSPECTIVE REFORMULATION

Towards Perspective Reformulation

Assume (for a moment) that

$$F(t) = a \cdot t + b \quad (a, b > 0),$$

so the congestion term in the objective function

$$\sum_{j \in J} v_j F(v_j) = a \sum_{j \in J} v_j^2 + b \sum_{j \in J} v_j$$

where

$$v_j = \sum_{i \in I} d_i x_{ij} \quad \forall j \in J$$

MINLP - rewritten in terms of v_j's

$$\min \sum_{j\in J} f_j y_j + \sum_{i\in I}\sum_{j\in J} d_i c_{ij} x_{ij} + b\sum_{j\in J} v_j + a\sum_{j\in J} v_j^2 \quad (1)$$

$$\text{s.t.} \quad \sum_{j\in J} y_j = p \quad (2)$$

$$\sum_{i\in I} d_i x_{ij} = v_j \qquad \forall j \in J \quad (3)$$

$$\sum_{j\in J} x_{ij} = 1 \qquad \forall i \in I \quad (4)$$

$$0 \leq x_{ij} \leq y_j \qquad \forall i \in I, j \in J \quad (5)$$

$$v_j \leq s_j y_j \qquad \forall j \in J \quad (6)$$

$$y_j \in \{0,1\} \qquad \forall j \in J \quad (7)$$

v_j are semi-continuous: $y_j = 0 \implies v_j = 0,$

$y_j = 1 \implies v_j \leq s_j.$

Perspective Reformulation

Replace v_j^2 by z_j in the OF and make a second-order constraint (SOC)

$$\min \sum_{j \in J} f_j y_j + \sum_{i \in I} \sum_{j \in J} d_i c_{ij} x_{ij} + b \sum_{j \in J} v_j + a \sum_{j \in J} z_j$$

$$\text{s.t.} \quad \sum_{j \in J} y_j = p$$

$$\sum_{i \in I} d_i x_{ij} = v_j \qquad \forall j \in J$$

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$$y \in \{0,1\}^{|J|}, z \geq 0$$

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$$v_j \le s_j y_j \qquad \forall j \in J$$

$$v_j^2 \le z_j y_j \qquad \forall j \in J \qquad \text{(SOC)}$$

$$y \in \{0,1\}^{|J|}, z \ge 0$$

STEP 2: (GENERALIZED) BENDERS DECOMPOSITION

Benders Reformulation

$$\min \sum_{j \in J} f_j y_j + w \tag{8}$$

$$\text{s.t.} \quad w \geq \Phi(y) \tag{9}$$

$$\sum_{j \in J} y_j = p$$

$$\sum_{j \in J} s_j y_j \geq \sum_{i \in I} d_i \tag{10}$$

$$y \in \{0, 1\}^{|J|} \tag{11}$$

$\Phi(y)$ is convex: allocation plus congestion costs for a given y.

Variables x_{ij}, v_j, z_j projected out and replaced by a single w.

Constraints (10) ensure feasibility for any fixed value of y^*.

Geoffrion (1972) proposed a **generalized Benders decomposition** to solve such problems.

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Benders Subproblem: convex NLP for a fixed y

If we fix the value of y, the problem becomes a convex NLP:

$$
\begin{aligned}
\Phi(y) = \min \sum_{i \in I} \sum_{j \in J} d_i c_{ij} x_{ij} + b \sum_{j \in J} v_j + a \sum_{j \in J} z_j & \\
\text{s.t.} \quad \sum_{i \in I} d_i x_{ij} = v_j & \qquad \forall j \in J \\
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v_j \leq s_j y_j & \qquad \forall j \in J \\
v_j^2 \leq z_j y_j & \qquad \forall j \in J \\
z \geq 0 &
\end{aligned}
$$

For a fixed (possibly fractional) value of y:
SOCP turned into a QCP. UBs on x and v variables.
Use your favorite NLP solver to find $\Phi(y)$ (e.g., CPLEX).

STEP 3: BRANCH-AND-CUT

Generalized Benders Decomposition

In a more general setting, we have

$$
\begin{aligned}
(P) \qquad & \min f(x, y) \\
& \text{s.t. } g(x, y) \leq 0 \\
& \qquad\qquad y \in Y
\end{aligned}
$$

Functions f and g are convex, y are *complicating* (integer) variables.
Benders reformulation:

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\begin{aligned}
(B) \qquad & \min w \\
& \text{s.t. } w \geq \Phi(y) \qquad \text{(OCuts)} \\
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Φ is the **value function**: $\Phi(\hat{y}) = \min_x \{f(x, \hat{y}) \mid g(x, \hat{y}) \leq 0\}$

$\Phi(y)$ is convex in y

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Generalized Benders Decomposition: Idea

Benders reformulation:

$$(B) \qquad \min w$$
$$\text{s.t. } w \geq \Phi(y) \qquad \text{(OCuts)}$$
$$y \in Y$$

Relaxed Master Problem (RMP)

$$\min w$$
$$y \in Y, w \geq 0$$

Benders separation:

1. Let (y^*, w^*) be the optimal solution of RMP.
2. Check if $w^* \geq \Phi(y^*)$. If not, add violated optimality cut to RMP.
3. Resolve RMP.

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The function $\Phi(y)$ is underestimated by tangential hyperplanes:

$$\Phi(y) \geq \Phi(y^*) + \alpha^T(y - y^*)$$

where α is a subgradient of Φ in y^*.

Benders optimality cut:

$$w \geq \Phi(y^*) + \alpha^T(y - y^*)$$

Implementation:

- Old School: Resolve RMP as a MIP. **Caveat:** each new cut requires solving the RMP as a MIP!
- Modern Benders: Remove integrality requirements from the RMP and embed it into a B&B $\Rightarrow$ Branch-and-Cut! Inserted cuts are globally valid!

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How to find α?[1]

Reformulate Benders subproblem $\Phi(y^*) = \min\{f(x, y^*) : g(x, y^*) \leq 0\}$ as

$$
\begin{aligned}
(S) \qquad & \min f(x, q) \\
& \text{s.t. } g(x, q) \leq 0 \\
& \qquad \text{s.t. } y^* \leq q \leq y^*
\end{aligned}
\tag{12}
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Then, in particular, a subgradient α is the reduced cost vector w.r.t. variables q. So, the optimality cut:

$$w \geq \Phi(y^*) + \alpha^T (y - y^*)$$

can be derived without explicitly invoking the computation of Lagrangian dual multipliers and subgradients of f and g.

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[1] By Lagrangian duality: $\Phi(y) = f(x, y) + \lambda^T g(x, y)$, and

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COMPUTATIONAL RESULTS

Computational Settings

- Comparing our generalized Benders decomposition framework with the perspective reformulation
- IBM ILOG Cplex 12.6.1
- Cluster: Intel Xeon E3-1220V2 @ 3.1GHz, with 16GB of RAM, 4 threads.
- Timelimit: 50,000 seconds.
- `CPX_PARAM_EPGAP`=1e-6
- Branch-and-cut:
  - Stabilization at the root node (multi-thread)
  - ≤ 100 cuts at the root, ≤ 20 at the nodes
  - Restart the root node twice (static Benders cuts & incumbent enforce variable fixing, internal Cplex cuts)

Benchmark Instances

- i*: Used to test the branch-and-cut-and-price algorithm by Avella & Boccia (2009) for linear CFL.
- Available at http://www.ing.unisannio.it/boccia/CFLP.htm.
- Generated following the procedure proposed in Cornuéjols, Sridharan, Thizy (1991) for linear CFL.
- 100 instances of size $|J| \times |I| \in \{300 \times 300, 300 \times 1500, 500 \times 500, 700 \times 700, 1,000 \times 1,000\}$ and $r \in \{5, 10, 15, 20\}$.
- Congestion function $a = b = 0.75$ (as suggested by Desrochers et al.)
- $p = \lfloor \pi |J| \rfloor$, where $\pi \in \{0.4, 0.6, 0.8\}$.

Instances 300 × 300

inst.	$\lvert J \rvert$	$\lvert I \rvert$	π	OPT	Perspective Reformulation (Cplex) $gap_r[\%]$	$t_r[s]$	$gap[\%]$	$t[s]$	nodes	Benders B&C $gap_r[\%]$	$t_r[s]$	$gap[\%]$	$t[s]$	nodes
1	300	300	0.4	257315.7360	0.1186	149	0.0000	233	130	0.0031	319	0.0000	383	10
1	300	300	0.6	214609.8293	0.0050	5057	0.0000	25115	17	0.0050	373	0.0000	594	30
1	300	300	0.8	219221.1886	0.4101	80	0.0000	155	49	0.0001	226	0.0000	236	0
6	300	300	0.4	273308.0002	0.0044	5675	0.0000	36598	28	0.0056	347	0.0000	583	32
6	300	300	0.6	225383.3635	0.0015	2329	0.0000	11696	9	0.0015	290	0.0000	299	3
6	300	300	0.8	228139.1799	0.0006	1825	0.0000	6196	5	0.0006	144	0.0000	165	0
11	300	300	0.4	259294.8204	0.0019	4936	0.0000	26260	17	0.0020	257	0.0000	271	3
11	300	300	0.6	216415.5633	0.0001	2257	0.0000	2257	0	0.0001	159	0.0000	167	0
11	300	300	0.8	224171.5836	0.0007	1842	0.0000	5273	5	0.0005	248	0.0000	254	0
16	300	300	0.4	256734.9041	0.0011	7417	0.0000	44495	33	0.0012	230	0.0000	281	8
16	300	300	0.6	220241.5213	0.0018	3063	0.0000	9463	6	0.0016	238	0.0000	250	3
16	300	300	0.8	231623.9657	0.0001	2888	0.0000	4641	3	0.0001	106	0.0000	115	0

All instances of size 300 × 300 solved to optimality in less than 10 minutes.

Instances 500 × 500

inst.	$\|J\|$	$\|I\|$	π	OPT	Perspective Reformulation (Cplex) $gap_r[\%]$	$t_r[s]$	$gap[\%]$	$t[s]$	nodes	Benders B&C $gap_r[\%]$	$t_r[s]$	$gap[\%]$	$t[s]$	nodes
1	500	500	0.4	433836.6527	2.3892	667	0.0000	1311	294	0.0017	1265	0.0000	2282	32
1	500	500	0.6	361323.3070	0.0006	50130	0.0006	TL	0	0.0007	1259	0.0000	1435	5
1	500	500	0.8	368022.2916	0.0000	17071	0.0000	17071	0	0.0000	587	0.0000	587	0
6	500	500	0.4	465717.4928	0.0005	4255	0.0005	TL	4	0.0005	823	0.0000	901	5
6	500	500	0.6	384364.0093	0.0004	11594	0.0004	TL	4	0.0006	975	0.0000	1430	12
6	500	500	0.8	393072.8259	0.0001	3000	0.0000	19734	3	0.0001	506	0.0000	532	0
11	500	500	0.4	420862.4354	0.0025	11074	0.0025	TL	5	0.0028	971	0.0000	2027	133
11	500	500	0.6	353185.6836	0.0009	3296	0.0009	TL	6	0.0010	962	0.0000	1664	32
11	500	500	0.8	366081.6812	0.0001	2698	0.0000	2698	0	0.0001	768	0.0000	814	0
16	500	500	0.4	398241.0995	0.0009	7310	0.0009	TL	5	0.0009	766	0.0000	946	9
16	500	500	0.6	345164.2574	0.0008	4956	0.0008	TL	5	0.0008	1061	0.0000	1719	17
16	500	500	0.8	367401.3250	0.0001	2984	0.0000	19396	3	0.0001	589	0.0000	639	0

Benders: All instances of size 500 × 500 solved to optimality in less than 40 minutes.

Cplex: in more than 50% of the cases reaches the TL (50 000 sec.s)

Instances 700 × 700

inst.	$\|J\|$	$\|I\|$	π	OPT	Perspective Reformulation (Cplex) gap_r[%]	$t_r[s]$	gap[%]	$t[s]$	nodes	Benders B&C gap_r[%]	$t_r[s]$	gap[%]	$t[s]$	nodes
1	700	700	0.4	608104.4400	0.4047	1760	0.0000	3106	345	0.0007	2306	0.0000	4485	128
1	700	700	0.6	511852.0282	0.0002	49519	0.0002	TL	1	0.0003	2495	0.0000	2532	2
1	700	700	0.8	528832.2478	0.0005	11270	0.0005	TL	7	0.0005	2274	0.0000	3117	6
6	700	700	0.4	590223.9309	0.0009	17338	0.0009	TL	5	0.0009	2126	0.0000	3093	20
6	700	700	0.6	491995.9402	0.0001	12952	0.0001	TL	7	0.0002	2196	0.0000	2600	4
6	700	700	0.8	512486.5658	0.0001	11506	0.0000	11506	0	0.0001	1195	0.0000	1307	0
11	700	700	0.4	588518.4248	0.0017	50117	0.0017	TL	0	0.0021	2342	0.0000	19450	2630
11	700	700	0.6	496861.5248	0.0011	12357	0.0011	TL	7	0.0012	2462	0.0000	11539	528
11	700	700	0.8	514897.8446	0.0005	11357	0.0003	TL	8	0.0005	2327	0.0000	3175	8
16	700	700	0.4	591092.0635	0.0010	15292	0.0010	TL	6	0.0012	2326	0.0000	5530	382
16	700	700	0.6	498257.7984	0.0004	14950	0.0004	TL	6	0.0004	2002	0.0000	2294	5
16	700	700	0.8	515610.2532	0.0001	11468	0.0000	TL	8	0.0001	1227	0.0000	1332	0

Benders: All instances of size 700 × 700 solved to optimality in less than 20 000 sec.s (most in about 1h).

Cplex: in more than 80% of the cases reaches the TL (50 000 sec.s)

Instances 1000 × 1000

inst.	$\lvert J\rvert$	$\lvert I\rvert$	π	OPT	Perspective Reformulation (Cplex) gap_r[%]	$t_r[s]$	gap[%]	$t[s]$	nodes	Benders B&C gap_r[%]	$t_r[s]$	gap[%]	$t[s]$	nodes
1	1000	1000	0.4	831618.2005	1.0782	5785	0.0000	12826	705	0.0015	7245	0.0006	TL	3649
1	1000	1000	0.6	700140.6641	–	–	–	–	–	0.0006	6102	0.0000	19435	237
1	1000	1000	0.8	720445.3031	–	–	–	–	–	0.0004	8256	0.0000	11922	7
6	1000	1000	0.4	884498.8703	–	–	–	–	–	0.0007	5849	0.0000	10162	90
6	1000	1000	0.6	739680.3837	–	–	–	–	–	0.0002	6640	0.0000	10429	13
6	1000	1000	0.8	765867.8192	–	–	–	–	–	0.0002	6829	0.0000	7263	2
11	1000	1000	0.4	808297.2103	–	–	–	–	–	0.0012	6003	0.0001	TL	2666
11	1000	1000	0.6	692675.4305	–	–	–	–	–	0.0003	4907	0.0000	8085	22
11	1000	1000	0.8	729765.8357	–	–	–	–	–	0.0002	5631	0.0000	6353	3
16	1000	1000	0.4	852614.2315	–	–	–	–	–	0.0015	4697	0.0005	TL	4700
16	1000	1000	0.6	719272.2322	–	–	–	–	–	0.0003	5503	0.0000	9205	37
16	1000	1000	0.8	744746.7001	–	–	–	–	–	0.0001	3098	0.0000	3208	2

Benders: Most instances of size 1000 × 1000 solved to optimality within the TL.

Root relaxation: Benders ($<$ 2h) with extremely small gaps!

Cplex: even impossible to solve the initial continuous relaxation (MINLP with 1M of variables and 1000 SOC)

Conclusion

- Solving convex MINLP with branch-and-cut implementation of the generalized Benders decomposition
- Strong perspective reformulation improves the root relaxation
- Projecting out variables crucial: otherwise impossible to solve the continuous relaxation
- Even though the Benders subproblem is not separable, we draw advantage of decomposition for two reasons:
  - reduce the number of variables from $O(m \cdot n)$ to $O(m)$
  - the compact model is a mixed-integer NLP $\Rightarrow$ transformed into a MILP

- Further applications: congestion in transportation (convex flow-costs), multi-commodity network design, two-stage stochastic opt. with convex recourse...

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Thank you!

M. Fischetti, I. Ljubić, M. Sinnl:
Benders decomposition without separability: A computational study for capacitated facility location problems,
European Journal of Operational Research 253(3): 557-569, 2016.