From Game Theory to Graph Theory: A Bilevel Journey

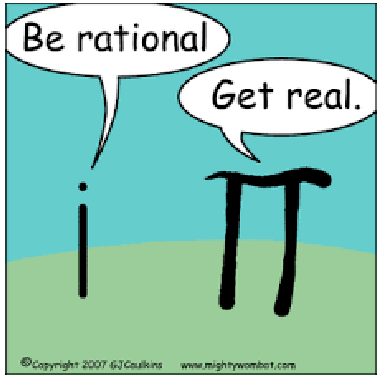
Ivana LJUBIC

ESSEC Business School, Paris

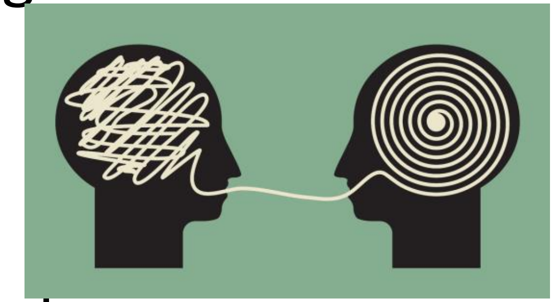
OR 2018, Brussels

EURO Plenary

Stackelberg Games



- **Two-player sequential-play game:** LEADER and FOLLOWER
- LEADER moves before FOLLOWER - first mover advantage
- **Perfect information:** both agents have perfect knowledge of each others strategy
- **Rationality:** agents act optimally, according to their respective goals
- LEADER takes FOLLOWERS's optimal response into account



- **Optimistic vs Pessimistic:** when FOLLOWER has multiple optimal responses

Stackelberg Games

- **Two-player sequential-play game:** LEADER and FOLLOWER

- LEADER moves ... FO...

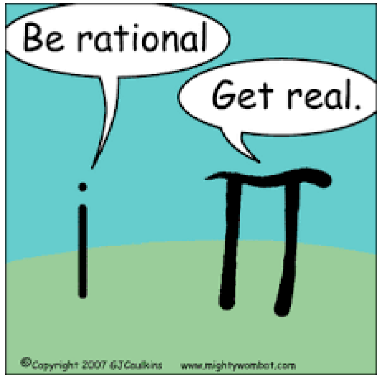
- **Perfec...** ...h others strategy

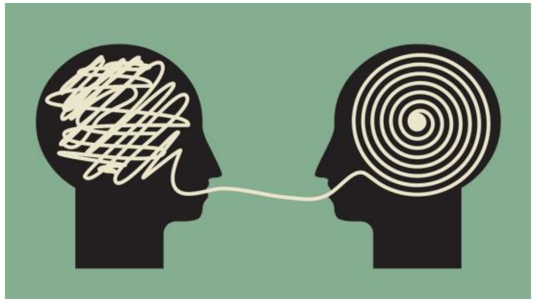
- **Ration...** ...goals
  - In ... which is the best according to hi...

- LEADER takes FOLLOWERS's opti... ...sponse into account

- **Optimistic vs Pessimistic:** when FOLLOWER has multiple optimal responses

STACKELBERG EQUILIBRIUM:
Find the best strategy for LEADER
(knowing what will be FOLLOWER‘s
best response)





Stackelberg Games

- Introduced in economy by v. Stackelberg in 1934
- 40 years later introduced in Mathematical Optimization → **Bilevel Optimization**

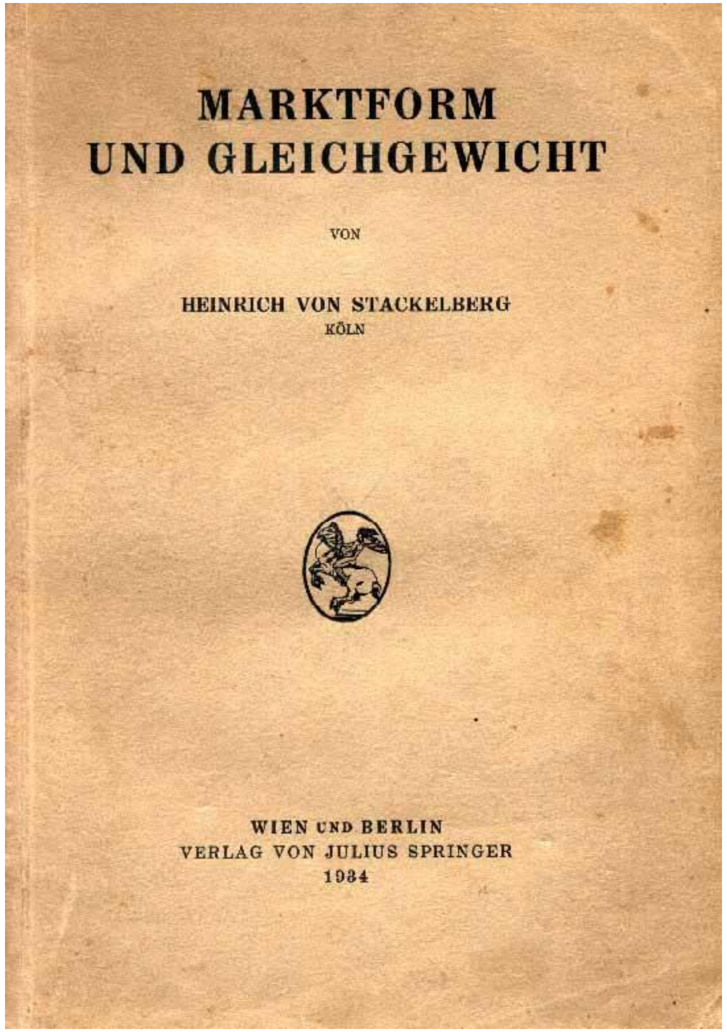
A Convex Programming Model for Optimizing SLBM Attack of Bomber Bases

Jerome Bracken and James T. McGill

Institute for Defense Analyses, Arlington, Virginia

(Received July 30, 1970)

This paper formulates a convex programming model allocating submarine-launched ballistic missiles (SLBMs) to launch areas and providing simultaneously an optimal targeting pattern against a specified set of bomber bases. Flight times of missiles from launch areas to bases vary and targets decrease in value over time. A nonseparable concave objective function is given for expected destruction of bombers. An example is presented.



Applications: Pricing

Two competitive agents act in a hierarchical way with different/conflicting objectives

- **Pricing:** operator sets tariffs, and then customers choose the cheapest alternative

- Tariff-setting, toll optimization (Labbé et al., 1998; Brotcorne et al., 2001)
- Network Design and Pricing (Brotcorne et al., 2008)
- Survey (van Hoesel, 2008)

Applications: Interdiction

Canada and the Transcontinental Drug Links

Strategic Forecasting Inc
go to original

Email Page   Print Page   Email Us

Canadian police conducted several simultaneous raids on suspected drug traffickers in Newfoundland and Quebec provinces Oct. 11, arresting two dozen people and seizing marijuana, cocaine, weapons, cash and property. The drug-trafficking ring, which Canadian authorities believe was operated by the Quebec-based Hell's Angels motorcycle/crime gang, could have smuggled the cocaine into Canada from South America via Mexico and the United States.

MAJOR DRUG SMUGGLING ROUTES THROUGH NORTH AMERICA

The jungles of South America, where cocaine is produced, seem a long way from the St. Lawrence River. Using a sophisticated shipment and distribution network, however, criminal and militant organizations can cover the distance in a few days.

More than 70 members of the Royal Newfoundland Constabulary and Quebec's Provincial Biker Enforcement Unit carried out the raids, which represented the culmination of an 18-monthlong investigation dubbed Operation Roadrunner. The arrests were made near St. John's in Newfoundland and near the towns of Laval and La Tuque in Quebec. In Newfoundland, authorities seized $300,000 in cash, 51 pounds of marijuana and 19 pounds of cocaine, as well as vehicles, weapons and computers. In Quebec, $170,000 and four houses were seized.

source: banderasnews.com

Applications: Interdiction

Canada and the Transcontinental Drug Links

Strategic Forecasting Inc
go to original

Email Page Print Page Email Us

Canadian police conducted several simultaneous raids on suspected drug traffickers in Newfoundland and Quebec provinces Oct. 11, arresting two dozen people and seizing marijuana, cocaine, weapons, cash and property. The drug-trafficking ring, which Canadian authorities believe was operated by the Quebec-based Hell's Angels motorcycle/crime gang, could have smuggled the cocaine into Canada from South America via Mexico and the United States.

The jungles of South America, where cocaine is produced, seem a long way from the St. Lawrence River. Using a sophisticated shipment and distribution network, however, criminal and militant organizations can cover the distance in a few days.

More than 70 members of the Royal Newfoundland Constabulary and Quebec's Provincial Biker Enforcement Unit carried out the raids, which represented the culmination of an 18-monthlong investigation dubbed Operation Roadrunner. The arrests were made near St. John's in Newfoundland and near the towns of Laval and La Tuque in Quebec. In Newfoundland, authorities seized $300,000 in cash, 51 pounds of marijuana and 19 pounds of cocaine, as well as vehicles, weapons and computers. In Quebec, $170,000 and four houses were seized.

source: banderasnews.com

- Monitoring / halting an adversary‘s activity on a network
  - Maximum-Flow Interdiction
  - Shortest-Path Interdiction
- Action:
  - Destruction of certain nodes / edges
  - Reduction of capacity / increase of cost on certain edges
- The problems are NP-hard! Survey (Collado and Papp, 2012)
- Uncertainties:
  - Network characteristics
  - Follower‘s response

Bilevel Optimization

General bilevel optimization problem

$$\min_{x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}} F(x, y)$$

$$G(x, y) \leq 0$$

$$y \in \arg \min_{y' \in \mathbb{R}^{n_2}} \{ f(x, y') : g(x, y') \leq 0 \}$$

Leader

Follower

Both players may involve integer decision variables, functions can be non-linear, non-convex…

Bilevel Optimization

General bilevel optimization problem

$$\min_{x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}} F(x, y)$$

$$G(x, y) \leq 0$$

$$y \in \arg\min_{y' \in \mathbb{R}^{n_2}} \{ f(x, y') : g(x, y') \leq 0 \}$$

Follower

1362 references!

Stephan Dempe

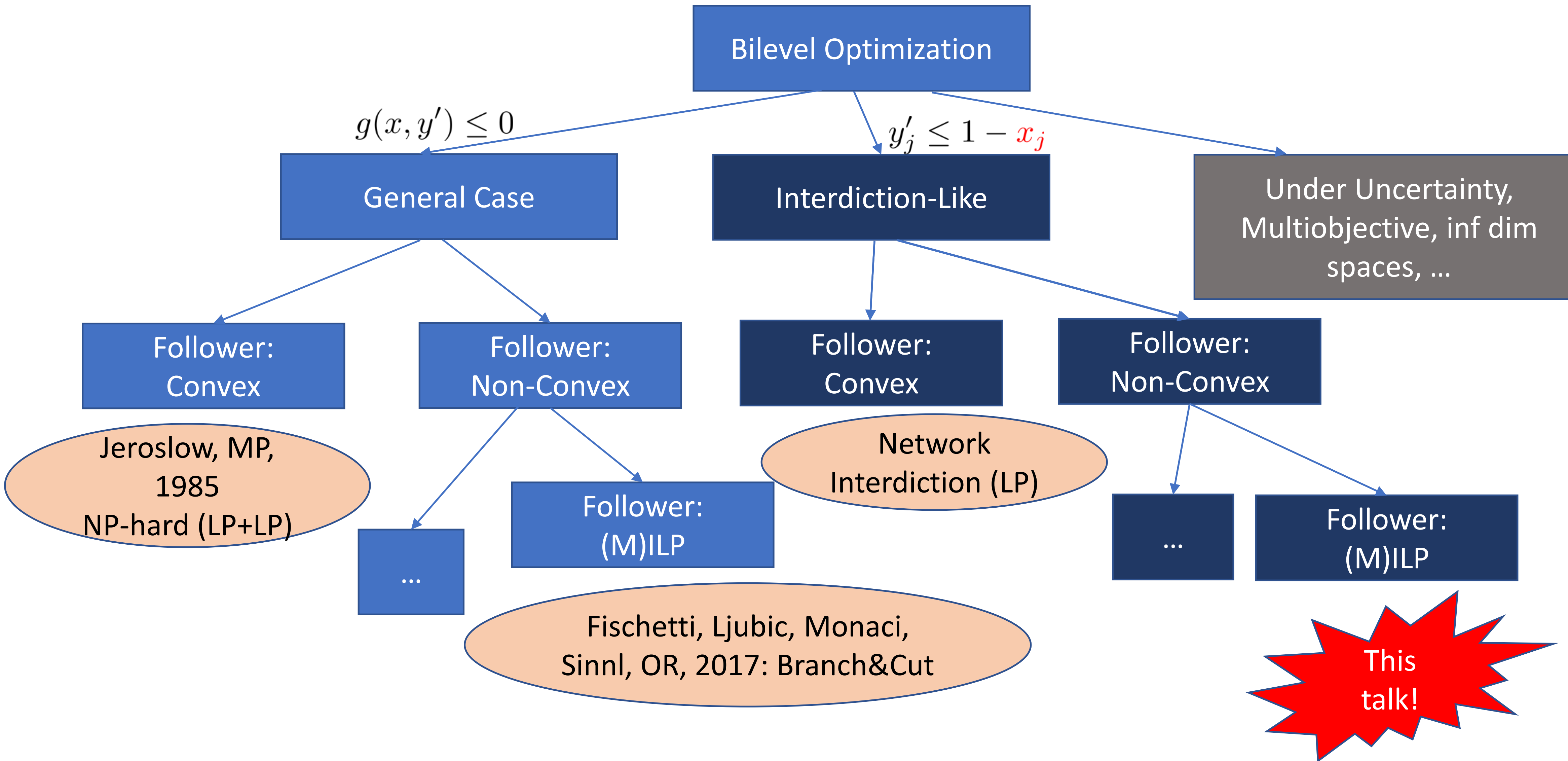
Bilevel optimization: theory, algorithms and applications

PREPRINT 2018-11

Fakultät für Mathematik und Informatik

integer decision variables, functions can be

Hierarchy of bilevel optimization problems



About our journey

- With **sparse MILP formulations**, we can now solve to optimality:
  - Covering Facility Location (Cordeau, Furini, Ljubic, 2018): **20M clients**
    - **Code:** https://github.com/fabiofurini/LocationCovering
  - Competitive Facility Location (Ljubic, Moreno, 2017): **80K clients (nonlinear)**
  - Facility Location Problems (Fischetti, Ljubic, Sinnl, 2016): 2K x 10K instances
  - Steiner Trees (DIMACS Challenge, 2014): 150k nodes, 600k edges

- **Common to all: Branch-and-Benders-Cut**

Is there a way to exploit sparse formulations along with Branch-and-Cut for bilevel optimization?

Problems addressed today…

- **Interdiction-Like Problems:** LEADER ”interdicts” FOLLOWER by removing some “objects”. Both agents play **pure strategies**.

- FOLLOWER solves a combinatorial optimization problem **(mostly, an NP-hard problem!).** One could build a payoff matrix (exponential in size!).

- We propose **a generic Branch-and-Interdiction-Cuts framework** to efficiently solve these problems in practice!
  - Assuming **monotonicty property** for FOLLOWER: **interdiction cuts** (facet-defining)
  - Computationally outperforming state-of-the-art

- Draw a connection to some problems in Graph Theory

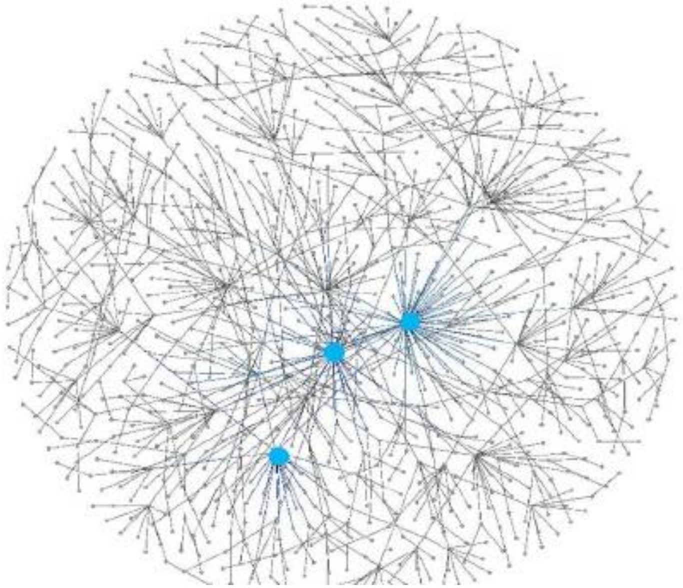
Based on a joint work with…

- M. Fischetti, I. Ljubic, M. Monaci, M. Sinnl: A new general-purpose algorithm for mixed-integer bilevel linear programs, *Operations Research* 65(6): 1615-1637, 2017
- M. Fischetti, I. Ljubic, M. Monaci, M. Sinnl: Interdiction Games and Monotonicity, with Application to Knapsack Problems, *INFORMS Journal on Computing*, in print, 2018
- F. Furini, I. Ljubic, P. San Segundo, S. Martin: The Maximum Clique Interdiction Game, *Optimization Online*, 2018
- F. Furini, I. Ljubic, E. Malaguti, P. Paronuzzi:
On Integer and Bilevel Formulations for the k-Vertex Cut Problem, submitted, 2018

Branch-and-Interdiction-Cut

A gentle introduction

Interdicting Communities in a Network

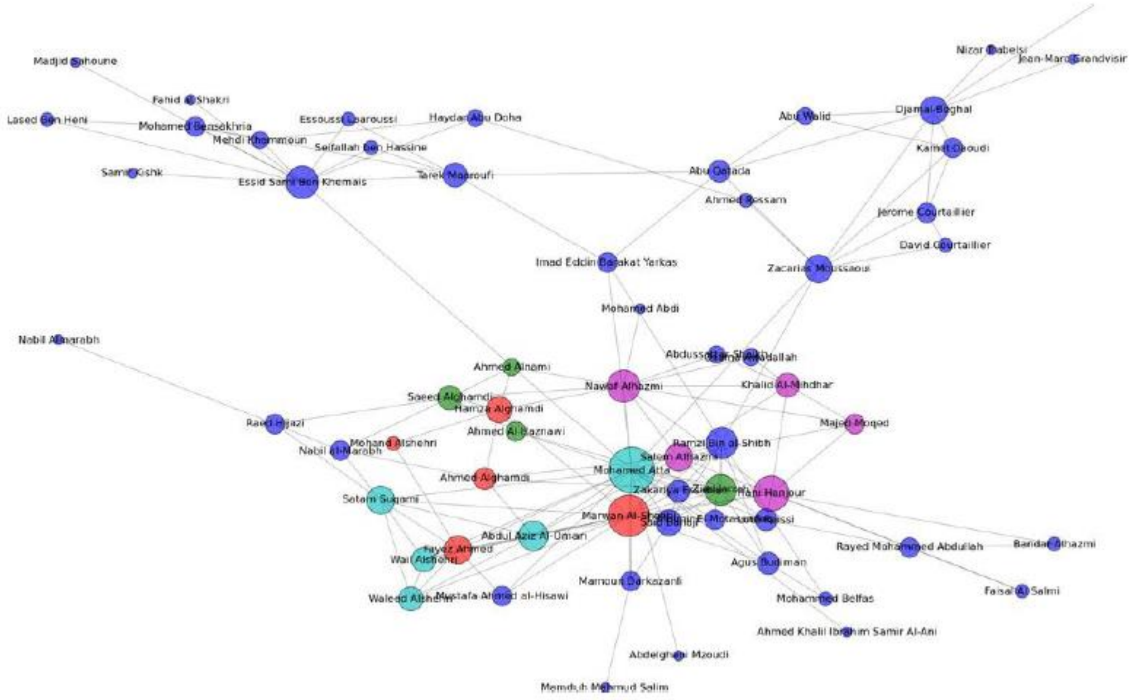


Critical Nodes: disconnect the network „the most“
Survey: Lalou et al. (2018)

Defender-Attacker Game

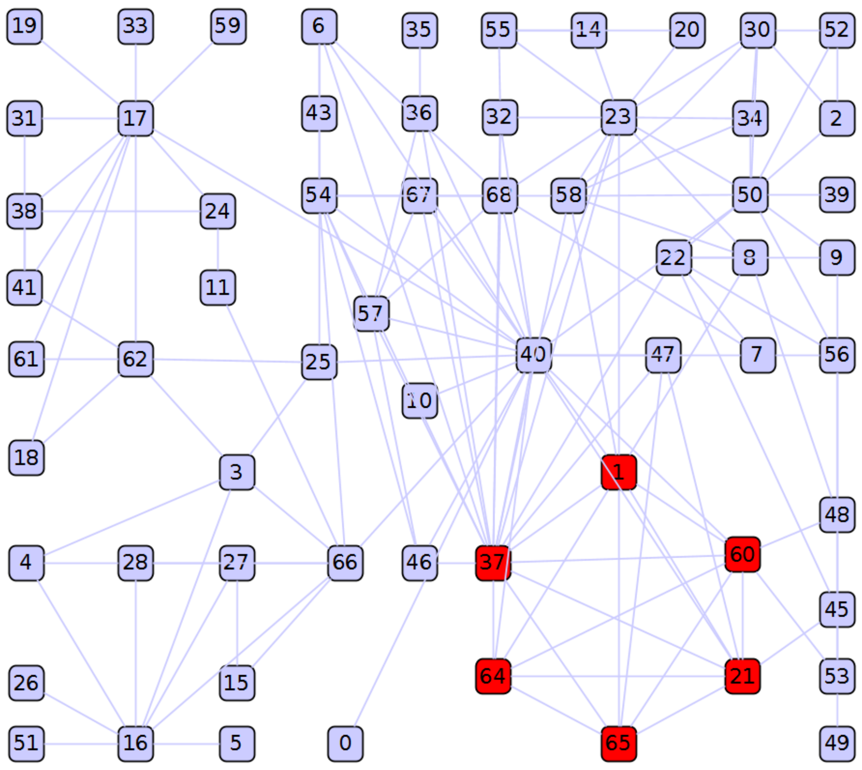
LEADER: eliminates the nodes
FOLLOWER: builds communities

After studying the lives of 172 terrorists, Sageman found the most common factors driving them are the social ties. Communities in social networks are often characterized as **densely connected subgraphs**.

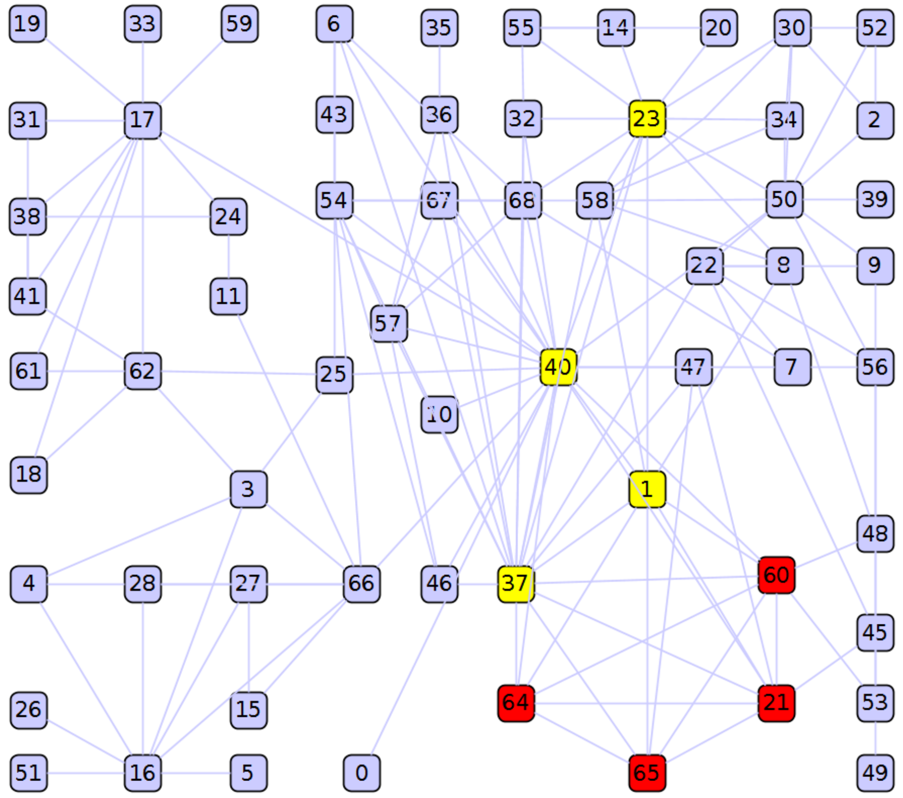


Research question: assuming that one can prevent k members of doing criminal activities, what is the size of the largest community that will remain?

Hamburg Cell: Max-Clique Interdiction

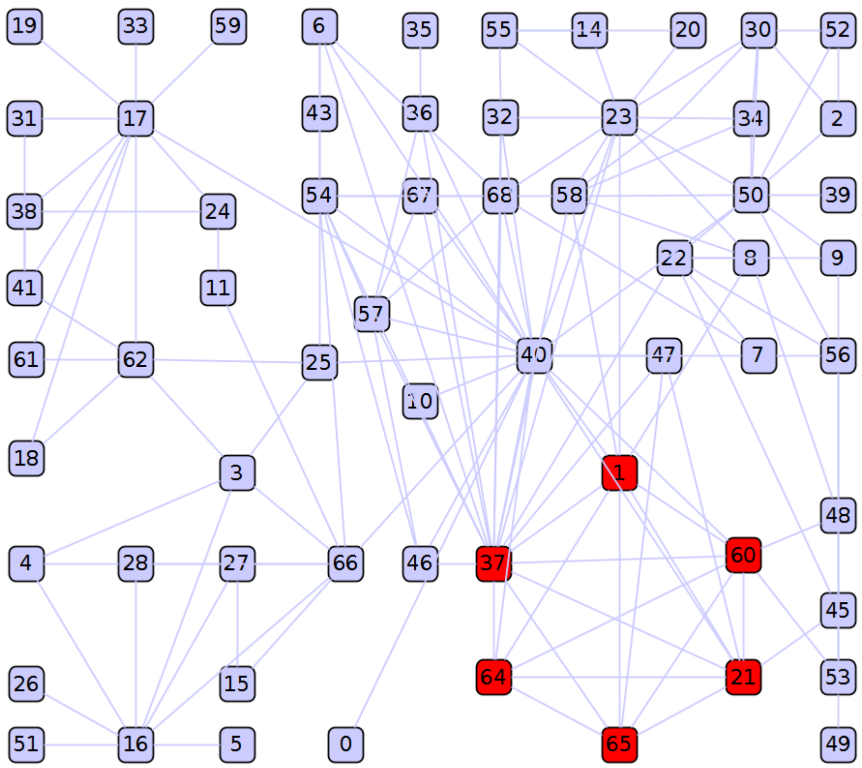


k=0

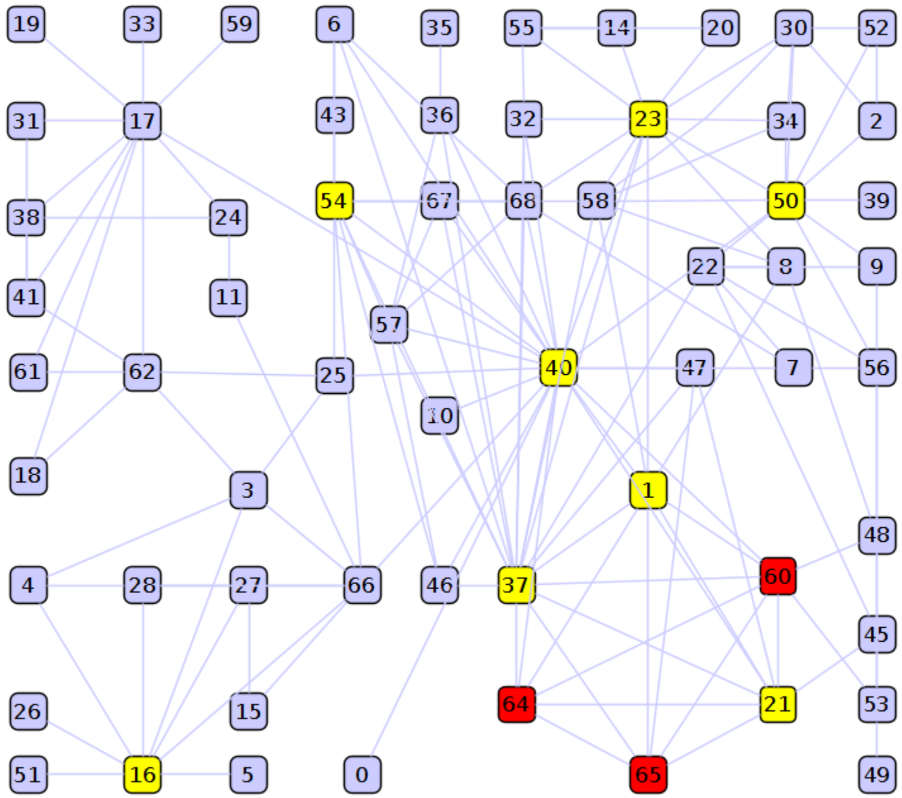


k=4

Hamburg Cell: Max-Clique Interdiction



k=0



k=8

Bilevel Integer Program

$$\min d^T y$$
$$b^T x \le B_D$$
$$y \in \arg\max\{d^T y : \quad y_i \le 1 - x_i, \quad i \in N \quad y \in Y\}$$
$$x_i \text{ binary}, \quad i \in N$$

$$\min w$$
$$b^T x \le B_D$$
$$w \ge \max\{d^T y : \quad y_i \le 1 - x_i, \quad i \in I \quad y \in Y\} = \Phi(\mathbf{x})$$
$$x_i \text{ binary}, \quad i \in I$$

Value Function

$$y_i = \begin{cases} 1 & \text{if node } i \text{ belongs to the community} \\ 0 & \text{otherwise} \end{cases} \quad i \in N.$$

$$x_i = \begin{cases} 1 & \text{if node } i \text{ is interdicted} \\ 0 & \text{otherwise} \end{cases} \quad i \in N.$$

Value Function Reformulation

$$\min_{x \in \mathbb{R}^{|N|}, w \in \mathbb{R}} \quad w$$

$$w \geq \Phi(x)$$

$$b^T x \leq B_D$$

$$x_i \text{ binary}, \quad i \in N$$

INTERDICTION: Min-max

$$\min_{x \in \mathbb{R}^{|N|}} \quad b^T x$$

$$K \geq \Phi(x)$$

$$x_i \text{ binary}, \quad i \in N$$

BLOCKING: Min-num or Min-sum

Value Function Reformulation

$$\min_{x \in \mathbb{R}^{|N|}, w \in \mathbb{R}} \quad w$$
$$w \geq \Phi(x)$$
$$b^T x \leq B_D$$
$$x_i \text{ binary}, \quad i \in N$$

$$\min_{x \in \mathbb{R}^{|N|}} \quad b^T x$$
$$K \geq \Phi(x)$$
$$x_i \text{ binary}, \quad i \in N$$

INTERDICTION: Min-max

BLOCKING: Min-num or Min-sum

GENERAL IDEA:

- Benders-Like Reformulation: y variables are projected out!
- If function $\Phi(x)$ could be "convexified" (using linear functions in x), we would obtain an MILP!
- To be solved in a branch-and-cut fashion

How to convexify the value function?

Convexification

Observation: Given x, for the optimal follower's response it holds:

$$x_j + y_j \le 1 \quad \Rightarrow \quad x_j y_j = 0 \qquad j \in N$$

Instead of solving:

$$\Phi(x) = \max_{y \in \mathbb{R}^{n_2}} d^T y$$
$$0 \le y_j \le 1 - x_j, \quad \forall j \in N$$
$$y \in Y$$

$$Y = \{y \in \mathbb{R}^{n_2} : \quad Q\, y \le q_0,$$
$$y_j \text{ integer } \forall j \in J_y\}.$$

Wood (2011) proposes to move x into the objective function and find the penalties M_j, such that we can equivalently solve:

$$\Phi(x) = \max_{y \in \mathbb{R}^{n_2}} \{d^T y - \sum_{j \in N} M_j x_j y_j \quad y \in Y\} \qquad = \max_{\hat{y} \in \mathrm{conv}(Y)} \{d^T \hat{y} - \sum_{j \in N} M_j x_j \hat{y}_j\}$$

Convexification → Benders-Like Reformulation

Benders–Like Reformulation

$$\min_{x \in \mathbb{R}^{n_1}, w \in \mathbb{R}} \quad w$$

$$w \geq d^T \hat{y} - \sum_{j \in N} M_j x_j \hat{y}_j \qquad \forall \hat{y} \in \hat{Y}$$

$$Ax \leq b$$

$$x_j \text{ integer}, \qquad \forall j \in J_x$$

$$x_j \text{ binary}, \qquad \forall j \in N.$$

The choice of M_j is crucial:

- If FOLLOWER solves an LP: Wood (2011), M_j is the upper bound of the dual variable.
- If FOLLOWER solves the KNAPSACK PROBLEM: Caprara et al. (2016), De Negre (2011), $M_j = d_j$.
- In general: **OPEN QUESTION**.

If the follower satisfies monotonicity property…

$$Y = \{y \in \mathbb{R}^{n_2} : Qy \leq q_0,$$
$$y_j \text{ integer } \forall j \in J_y\}.$$

Theorem (Fischetti, Ljubić, Monaci, Sinnl, 2018)
For Interdiction Games with Monotonicity $M_j = d_j$, i.e., we have:

$$\min_{x \in \mathbb{R}^{n_1}, w \in \mathbb{R}} w$$
$$w \geq \sum_{j \in N} d_j \hat{y}_j (1 - x_j) \qquad \forall \hat{y} \in \hat{Y}$$
$$Ax \leq b$$
$$x_j \text{ integer}, \qquad \forall j \in J_x$$
$$x_j \text{ binary}, \qquad \forall j \in N.$$

Downward Monotonicity: $Q \geq 0$

If $\hat{y}$ is a feasible follower and y' satisfies integrality constraints and $0 \leq y' \leq \hat{y}$, then y' is *also feasible*.

If the follower satisfies monotonicity property…

$$Y = \{y \in \mathbb{R}^{n_2} : Qy \le q_0,$$
$$y_j \text{ integer } \forall j \in J_y\}.$$

Theorem (Fischetti, Ljubić, Monaci, Sinnl, 2018)

For Interdiction Games with Monotonicity $M_j = d_j$, i.e., we have:

$$\min_{x \in \mathbb{R}^{n_1}, w \in \mathbb{R}} w$$
$$w \ge \sum_{j \in N} d_j \hat{y}_j (1 - x_j) \qquad \forall \hat{y} \in \hat{Y}$$
$$Ax \le b$$
$$x_j \text{ integer}, \qquad \forall j \in J_x$$
$$x_j \text{ binary}, \qquad \forall j \in N.$$

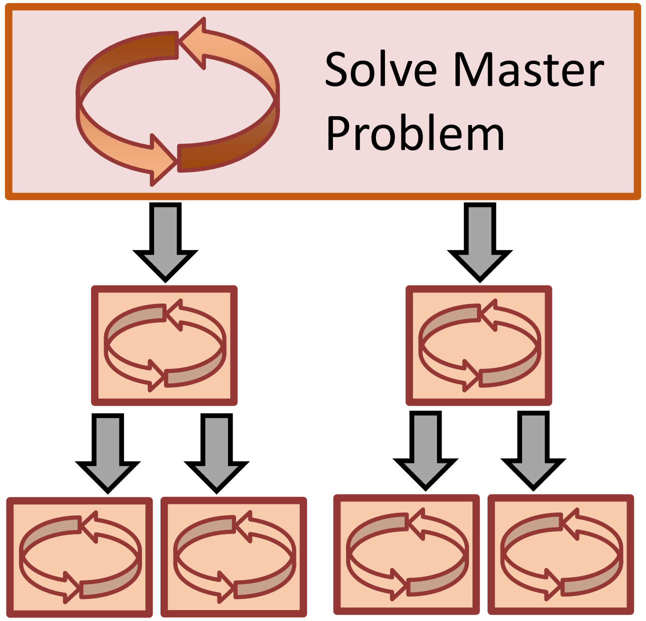
Downward Monotonicity: $Q \ge 0$

If $\hat{y}$ is a feasible follower and y' satisfies integrality constraints and $0 \le y' \le \hat{y}$, then y' is *also feasible*.

- **max–knapsack (set packing)**
- **max–clique**
- **max–relaxed–clique** (s**–plex: degree,** s**–clique: distance,** s**–bundle: connectivity)**

A Careful Branch-and-Interdiction-Cut Design

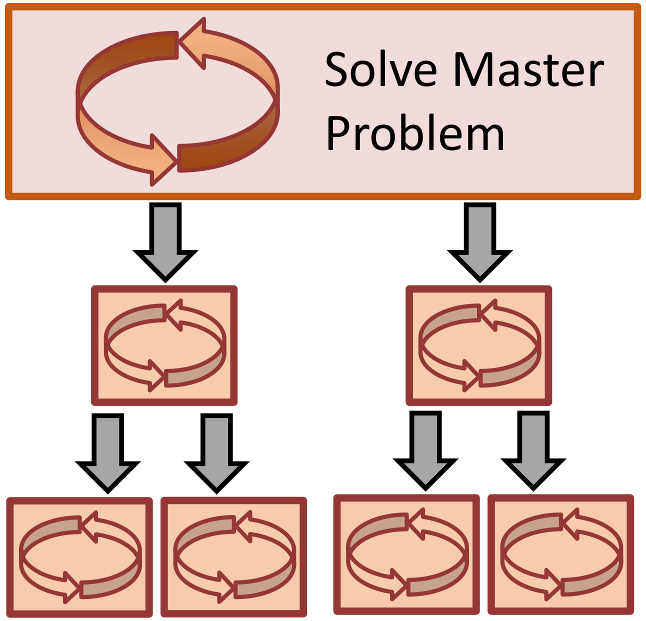
- **Separation:** finding the best FOLLOWER's response for a given x^*. **NP–hard, in general**.
- A good **balance** between "lazy cut separation" (integer points only) and "user cut separation" (fractional points).



$\rightarrow$ **Branch-and-Interdiction-Cut**

A Careful Branch-and-Interdiction-Cut Design

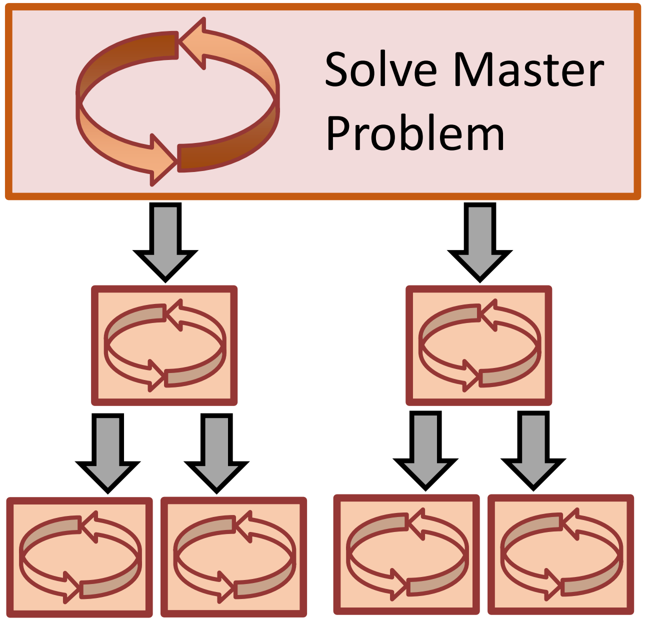
- **Separation:** finding the best FOLLOWER's response for a given x^*. **NP–hard, in general**.
- A good **balance** between "lazy cut separation" (integer points only) and "user cut separation" (fractional points).
- **Crucial:** **specialized procedures/algorithms** for FOLLOWER's sub-problem (if available).
- **Combinatorial** algorithms for **LOWER** and **UPPER BOUNDS**.
- Efficient **PREPROCESSING** techniques.



$\rightarrow$ **Branch-and-Interdiction-Cut**

A Careful Branch-and-Interdiction-Cut Design

- **Separation:** finding the best FOLLOWER's response for a given x^*. **NP–hard, in general**.
- A good **balance** between "lazy cut separation" (integer points only) and "user cut separation" (fractional points).
- **Crucial:** **specialized procedures/algorithms** for FOLLOWER's sub-problem (if available).
- **Combinatorial** algorithms for **LOWER** and **UPPER BOUNDS**.
- Efficient **PREPROCESSING** techniques.
- Under **monotonicity property**: Interdiction cuts are **facet–defining** or could be lifted, otherwise.
- Resulting in general in **strong LP–relaxation bounds**.



$\rightarrow$ **Branch-and-Interdiction-Cut**

Max-Clique-Interdiction on Large-Scale Networks

	Max-Clique Solver San Segundo et al. (2016)			$k = \lceil 0.005 \cdot \lvert V \rvert \rceil$		$k = \lceil 0.01 \cdot \lvert V \rvert \rceil$	
	$\lvert V \rvert$	$\lvert E \rvert$	ω [s]	t [s]	$\lvert V_p \rvert$	t [s]	$\lvert V_p \rvert$
socfb-UIllinois	30,795	1,264,421	0.5	24.4	10,456	41.6	8290
ia-email-EU	32,430	54,397	0.0	0.6	30,375	0.5	29,212
ia-enron-large	33,696	180,811	0.0	2.2	27,791	29.5	26,651
socfb-UF	35,111	1,465,654	0.3	17.8	14,264	87.8	10,708
socfb-Texas84	36,364	1,590,651	0.3	24.6	10,706	74.3	8,704
fe-tooth	78,136	452,591	0.5	18.9	7	19.0	7
sc-pkustk11	87,804	2,565,054	1.1	70.7	2,712	57.1	2,712
ia-wiki-Talk	92,117	360,767	0.2	49.2	72,678	87.4	72,678
sc-pkustk13	94,893	3,260,967	1.3	724.9	2,360	879.2	2,354

eliminated by preprocessing

Furini, Ljubic, Martin, San Segundo (2018)

Max-Clique-Interdiction on Large-Scale Networks

Max-Clique Solver
San Segundo et al. (2016)

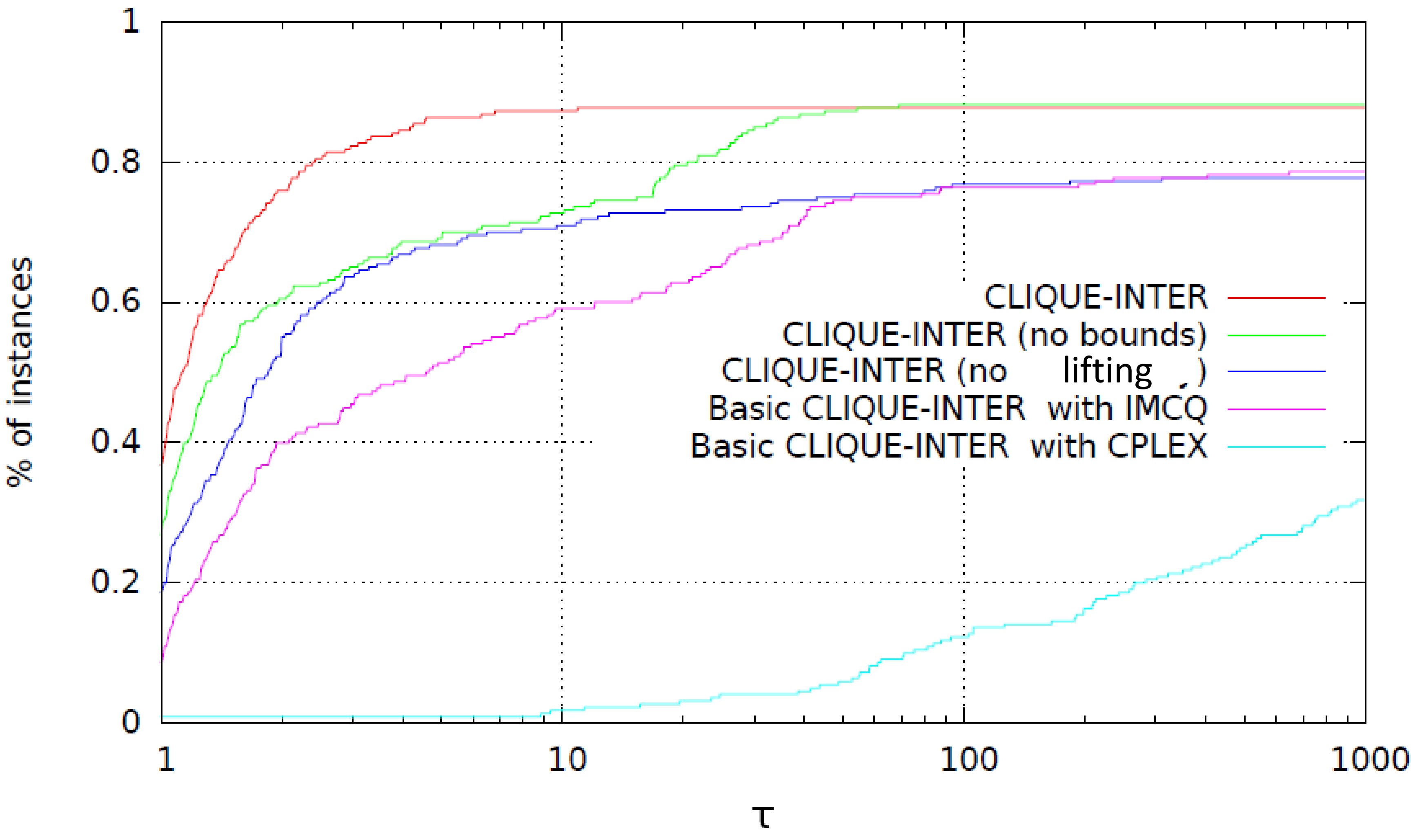
	$\lvert V\rvert$	$\lvert E\rvert$	ω [s]	$k = \lceil 0.005 \cdot \lvert V\rvert \rceil$ t [s]	$k = \lceil 0.005 \cdot \lvert V\rvert \rceil$ $\lvert V_p\rvert$	$k = \lceil 0.01 \cdot \lvert V\rvert \rceil$ t [s]	$k = \lceil 0.01 \cdot \lvert V\rvert \rceil$ $\lvert V_p\rvert$
socfb-UIllinois	30,795	1,264,421	0.5	24.4	10,456	41.6	8290
ia-email-EU	32,430	54,397	0.0	0.6	30,375	0.5	29,212
ia-enron-large	33,696	180,811	0.0	2.2	27,791	29.5	26,651
socfb-UF	35,111	1,465,654	0.3	17.8	14,264	87.8	10,708
socfb-Texas84	36,364	1,590,651	0.3	24.6	10,706	74.3	8,704
fe-tooth	78,136	452,591	0.5	18.9	7	19.0	7
sc-pkustk11	87,804	2,565,054	1.1	70.7	2,712	57.1	2,712
ia-wiki-Talk	92,117	360,767	0.2	49.2	72,678	87.4	72,678
sc-pkustk13	94,893	3,260,967	1.3	724.9	2,360	879.2	2,354

eliminated by preprocessing

#variables

Furini, Ljubic, Martin, San Segundo (2018)

B&IC Ingredients



Comparison with the state-of-the-art MILP bilevel solver

		Branch-and-Interdiction-Cut				Generic B&C for Bilevel MILPs (Fischetti, Ljubic, Monaci, Sinnl, 2017)			
$\|V\|$	#	# solved	time	exit gap	root gap	# solved	time	exit gap	root gap
50	44	**44**	**0.01**	-	0.16	28	68.58	6.44	8.50
75	44	**44**	**1.45**	-	0.41	14	120.19	9.47	10.91
100	44	**37**	**9.30**	1.00	0.98	7	164.42	12.65	13.11
125	44	**35**	**13.43**	1.33	1.20	2	135.33	13.88	14.73
150	44	**33**	**27.23**	1.91	1.43	1	397.52	16.42	16.39

Slide “NOT TO BE SHOWN”

The follower:

B&IC WORKS WELL EVEN
IF FOLLOWER HAS MORE
DECISION VARIABLES, AS LONG AS
MONOTONOCITY HOLDS FOR
INTERDICTED VARIABLES

Downward Monotonicity:

“if $\hat{y} = (\hat{y}_N, \hat{y}_R)$ is a feasible follower for a given x and $y' = (y'_N, \hat{y}_R)$ satisfies integrality constraints and $0 \leq y'_N \leq \hat{y}_N$, then y' is **also feasible** for x”.

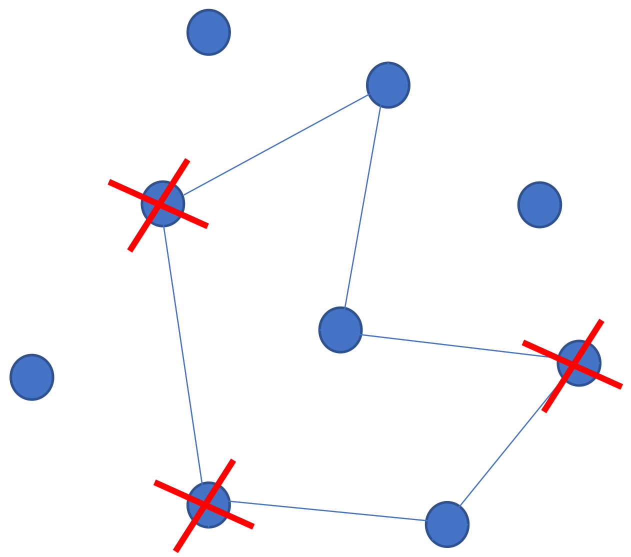
The result can be further generalized

Relevant Operations Research applications. Two companies competing at the market for customers.

- LEADER: established on the market,
- FOLLOWER: a newcomer who wants to disrupt the market.

LEADER wants to keep the customers by providing them coupons, vouchers. FOLLOWER is solving a profit-maximization problem:

- **NETWORK DESIGN**: prize-collecting Steiner tree
- **LOGISTICS**: orienteering problems
- **FACILITY LOCATION**: profit maximization variant

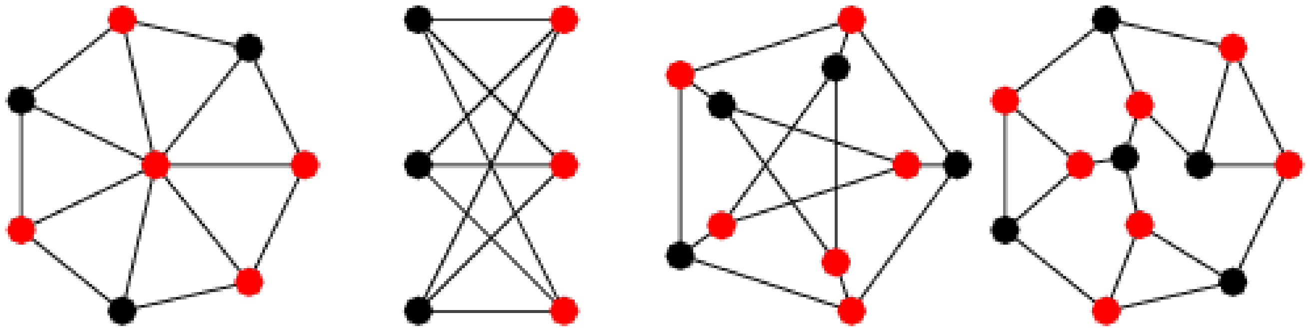


Fischetti, Ljubic, Monaci, Sinnl (2018)

And what about Graph Theory?

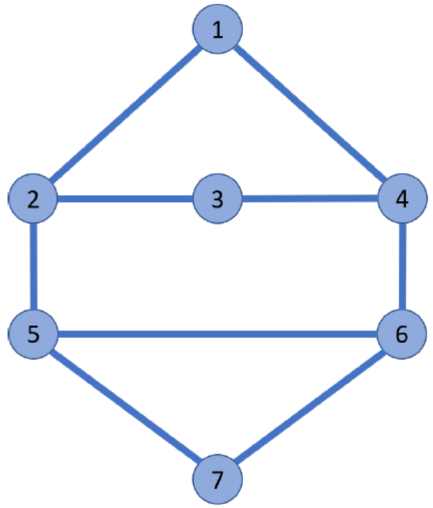
A weird example…

- **Property:** A set of vertices is a **vertex cover** if and only if its complement is an independent set
- **Vertex Cover as a Blocking Problem:**
  - LEADER: interdicts (removes) the nodes.
  - FOLLOWER: maximizes the size of the largest connected component in the remaining graph.
  - Find the smallest set of nodes to interdict, so that FOLLOWER's optimal response is at most one.



The k-Vertex-Cut Problem

- A set of vertices is a **vertex k–cut** if upon its removal the graph contains at least k components.
- **The k Vertex–Cut Problem:** Find a vertex k-cut of minimum cardinality/weight.

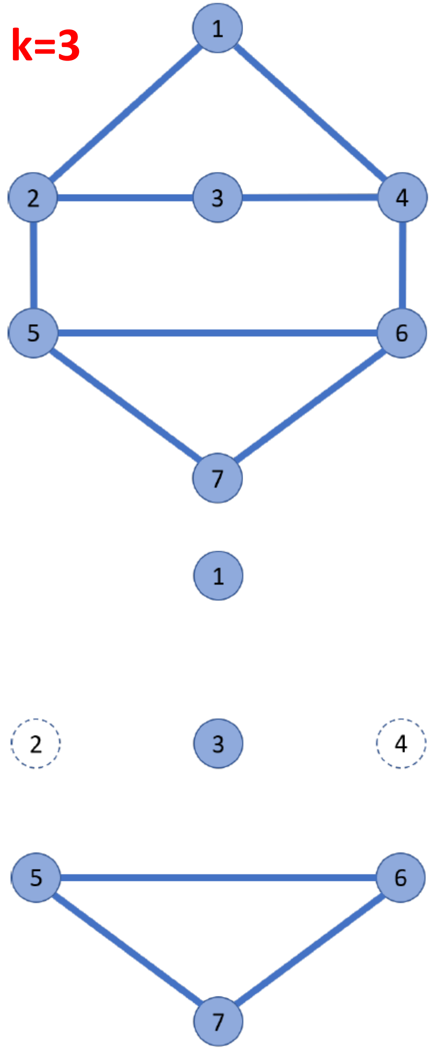


Open question: Is there an ILP formulation in the natural space of variables?

Furini, Ljubic, Malaguti, Paronuzzi (2018)

The k-Vertex-Cut Problem

- A set of vertices is a **vertex k–cut** if upon its removal the graph contains at least k components.
- **The k Vertex–Cut Problem:** Find a vertex k-cut of minimum cardinality/weight.
- Influential nodes in a diffusion model for social networks, Kempe et al. (2005)
- Decomposition method for linear equation systems, e.g. GCG solver (Bastubbe, Lübbecke, 2017)



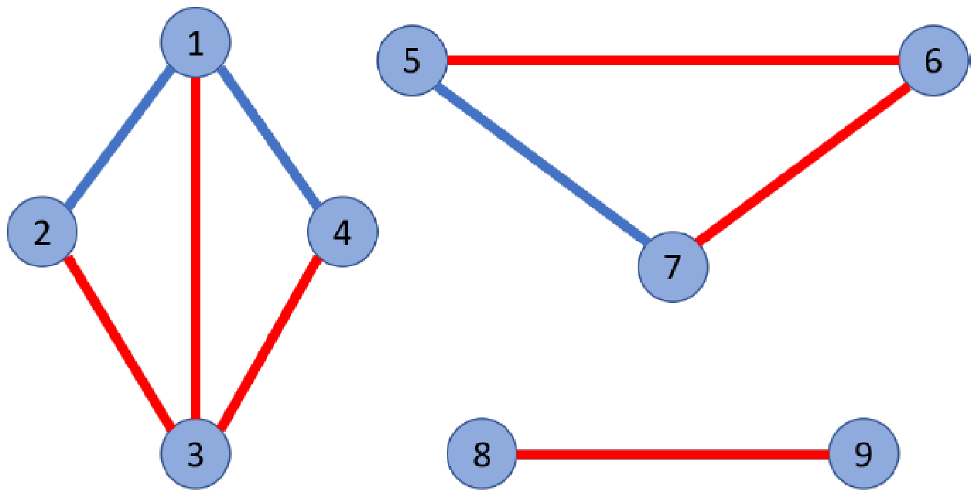
Open question: Is there an ILP formulation in the natural space of variables?

Furini, Ljubic, Malaguti, Paronuzzi (2018)

K-Vertex-Cut

Property: A graph G has at least k (not empty) components if and only if any cycle-free subgraph of G contains at most $|V| - k$ edges.

Example with $|V| = 9$ and $k = 3$:

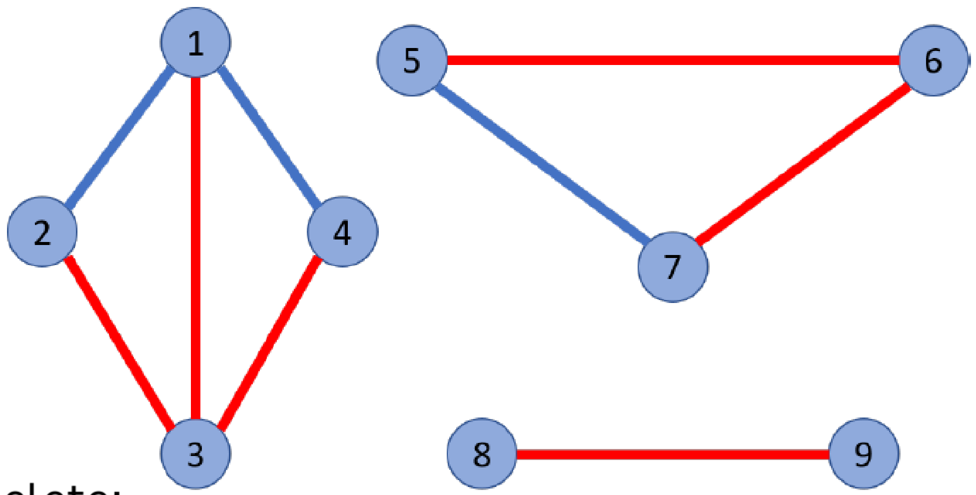


k=3

K-Vertex-Cut

Property: A graph G has at least k (not empty) components if and only if any cycle-free subgraph of G contains at most $|V| - k$ edges.

Example with $|V| = 9$ and $k = 3$:



Stackelberg game:

- LEADER: searches the smallest subset of nodes to delete;
- FOLLOWER maximizes the size of the cycle-free subgraph on the remaining graph.

k=3

k-Vertex-Cut: Benders-like reformulation

$$\min \sum_{v \in V} x_v$$

$$\Phi(x) \leq |V| - \sum_{v \in V} x_v - k$$

$$x_v \in \{0,1\} \qquad v \in V.$$

The value function reformulation

The following **Natural Space Formulation**, is a valid model for the k-vertex cut problem (Furini et al. 2018):

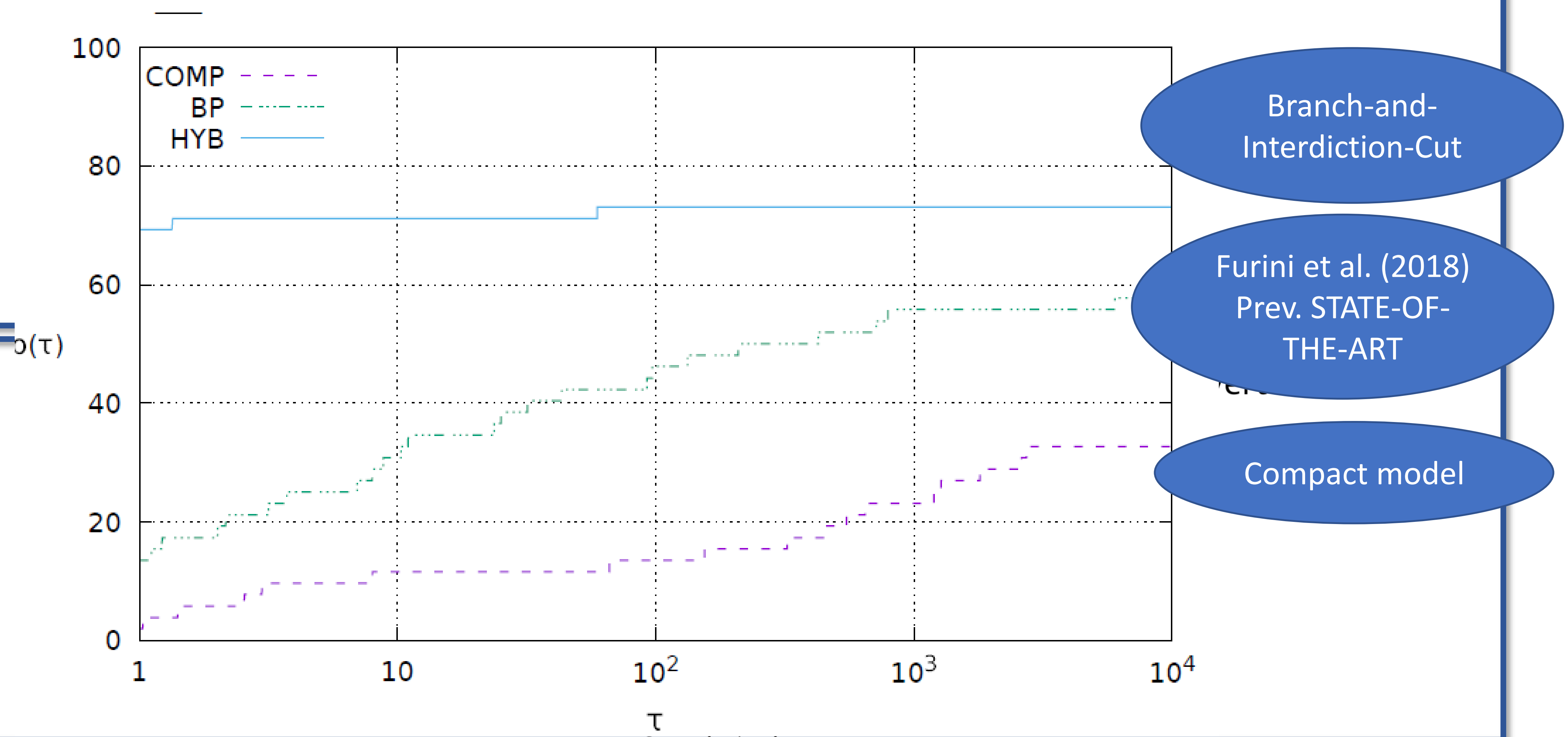
$$\min \sum_{v \in V} x_v$$

$$\sum_{v \in V} [\deg_T(v) - 1] x_v \geq k - |V| + |E(T)| \qquad T \in \mathcal{T},$$

$$x_v \in \{0,1\} \qquad v \in V.$$

Furini, Ljubic, Malaguti, Paronuzzi (2018)

k-Vertex-Cut: Benders-like reformulation



Furini, Ljubic, Malaguti, Paronuzzi (2018)

Conclusions.

And some directions for the future research.

Takeaways

- Bilevel optimization: very difficult!

- **Branch-and-Interdiction-Cuts** can work very well in practice:
  - Problem reformulation in the **natural space of variables** („**thinning out**“ the heavy MILP models)
  - **Tight** „**interdiction cuts**“ (**monotonicity** property)
  - **Crucial:** Problem-dependent (combinatorial) separation strategies, preprocessing, combinatorial poly-time bounds

- Many **graph theory problems** (node-deletion, edge-deletion) could be solved efficiently, when **approached from the bilevel-perspective**

Possible directions for future research

- **Bilevel Optimization:** a better way of **integrating customer behaviour** into decision making models

- Generalizations of **Branch-and-Interdiction-Cuts** for:
  - **Non-linear** follower functions
  - **Submodular** follower functions
  - Interdiction problems **under uncertainty**
  - …

- Extensions to **Defender-Attacker-Defender (DAD)** Models (**trilevel games**)

- **Benders-like decomposition for general mixed-integer bilevel optimization**

Thank you for your attention!

References:

- M. Fischetti, I. Ljubic, M. Monaci, M. Sinnl: A new general-purpose algorithm for mixed-integer bilevel linear programs, *Operations Research* 65(6): 1615-1637, 2017

  SOLVER: https://msinnl.github.io/pages/bilevel.html

- M. Fischetti, I. Ljubic, M. Monaci, M. Sinnl: Interdiction Games and Monotonicity, with Application to Knapsack Problems, *INFORMS Journal on Computing*, in print, 2018

- F. Furini, I. Ljubic, P. San Segundo, S. Martin: The Maximum Clique Interdiction Game, *Optimization Online*, 2018

- F. Furini, I. Ljubic, E. Malaguti, P. Paronuzzi:
  On Integer and Bilevel Formulations for the k-Vertex Cut Problem, submitted, 2018

Literature

- Bastubbe, M., Lübbecke, M.: A branch-and-price algorithm for capacitated hypergraph vertex separation. Technical Report, Optimization Online (2017)
- L. Brotcorne, M. Labbé, P. Marcotte, and G. Savard. A Bilevel Model for Toll Optimization on a Multicommodity Transportation Network, Transportation Science, 35(4): 345-358
- L. Brotcorne, M. Labbé, P. Marcotte, and G. Savard. Joint design and pricing on a network. Operations Research, 56 (5):1104–1115, 2008
- Caprara A, Carvalho M, Lodi A, Woeginger GJ (2016) Bilevel knapsack with interdiction constraints. INFORMS Journal on Computing 28(2):319–333
- R.A.Collado, D. Papp. Network interdiction – models, applications, unexplored directions, Rutcor Research Report 4-2012, 2012.
- J.F. Cordeau, F. Furini, I. Ljubic. Benders Decomposition for Very Large Scale Partial Set Covering and Maximal Covering Problems, submitted, 2018
- S. Dempe. Bilevel optimization: theory, algorithms and applications, TU Freiberg, ISSN 2512-3750. Fakultät für Mathematik und Informatik. PREPRINT 2018-11
- DeNegre S (2011) Interdiction and Discrete Bilevel Linear Programming. Ph.D. thesis, Lehigh University
- M. Fischetti, I. Ljubic, M. Sinnl: Redesigning Benders Decomposition for Large Scale Facility Location, Management Science 63(7): 2146-2162, 2017

Literature, cont.

- R.G. Jeroslow. The polynomial hierarchy and a simple model for competitive analysis. Mathematical Programming, 32(2):146–164, 1985
- Kempe, D., Kleinberg, J., Tardos, E.: Influuential nodes in a diffusion model for social networks. In: L. Caires, G.F. Italiano, L. Monteiro, C. Palamidessi, M. Yung (eds.) Automata, Languages and Programming, pp. 1127-1138. , 2005
- M. Labbé, P. Marcotte, and G. Savard. A bilevel model of taxation and its application to optimal highway pricing. Management Science, 44(12):1608–1622, 1998
- I. Ljubic, E. Moreno: Outer approximation and submodular cuts for maximum capture facility location problems with random utilities, European Journal of Operational Research 266(1): 46-56, 2018
- M. Sageman. Understanding Terror Networks. ISBN: 0812238087, University of Pennsylvania Press, 2005
- San Segundo P, Lopez A, Pardalos PM. A new exact maximum clique algorithm for large and massive sparse graphs. Computers & OR 66:81–94, 2016
- S. van Hoesel. An overview of Stackelberg pricing in networks. European Journal of Operational Reseach, 189:1393–1492, 2008
- R.K. Wood. Bilevel Network Interdiction Models: Formulations and Solutions, John Wiley & Sons, Inc., http://hdl.handle.net/10945/38416, 2010