Solving Very Large Scale Covering Location Problems using Branch-and-Benders-Cuts

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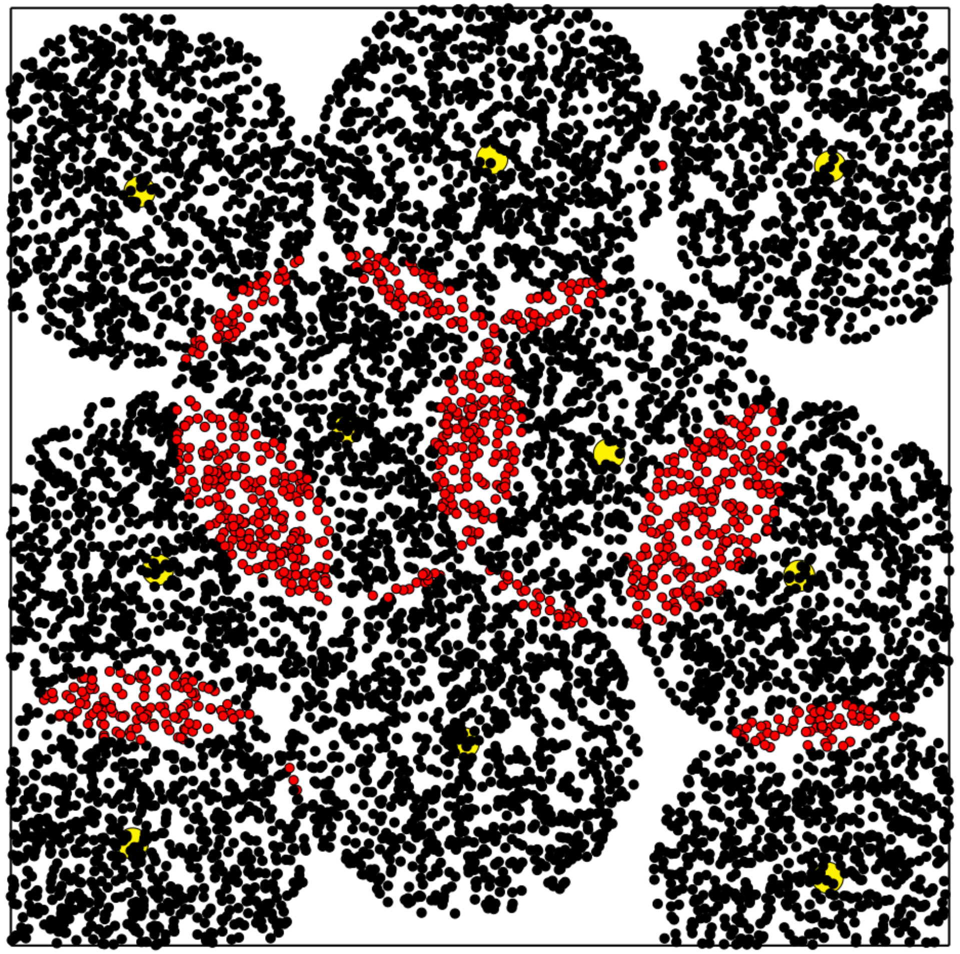
IWOLOCA 2019, Cadiz, Spain, February 1, 2019

Covering Location Problems

- Given:
  - Set of demand points (clients): **J**
  - Set of potential facility locations: **I**
  - A demand point is **covered** if it is within a neighborhood of at least one open facility

- **Set Covering Location Problem** (SCLP):
  - Choose the min-number of facilities to open so that each client is covered

  - Might be too restrictive
  - Gives the same importance to every point, regardless its position and size



Two Variants Studied in This Work

Additional input:

Demand d_j, for each client j from J

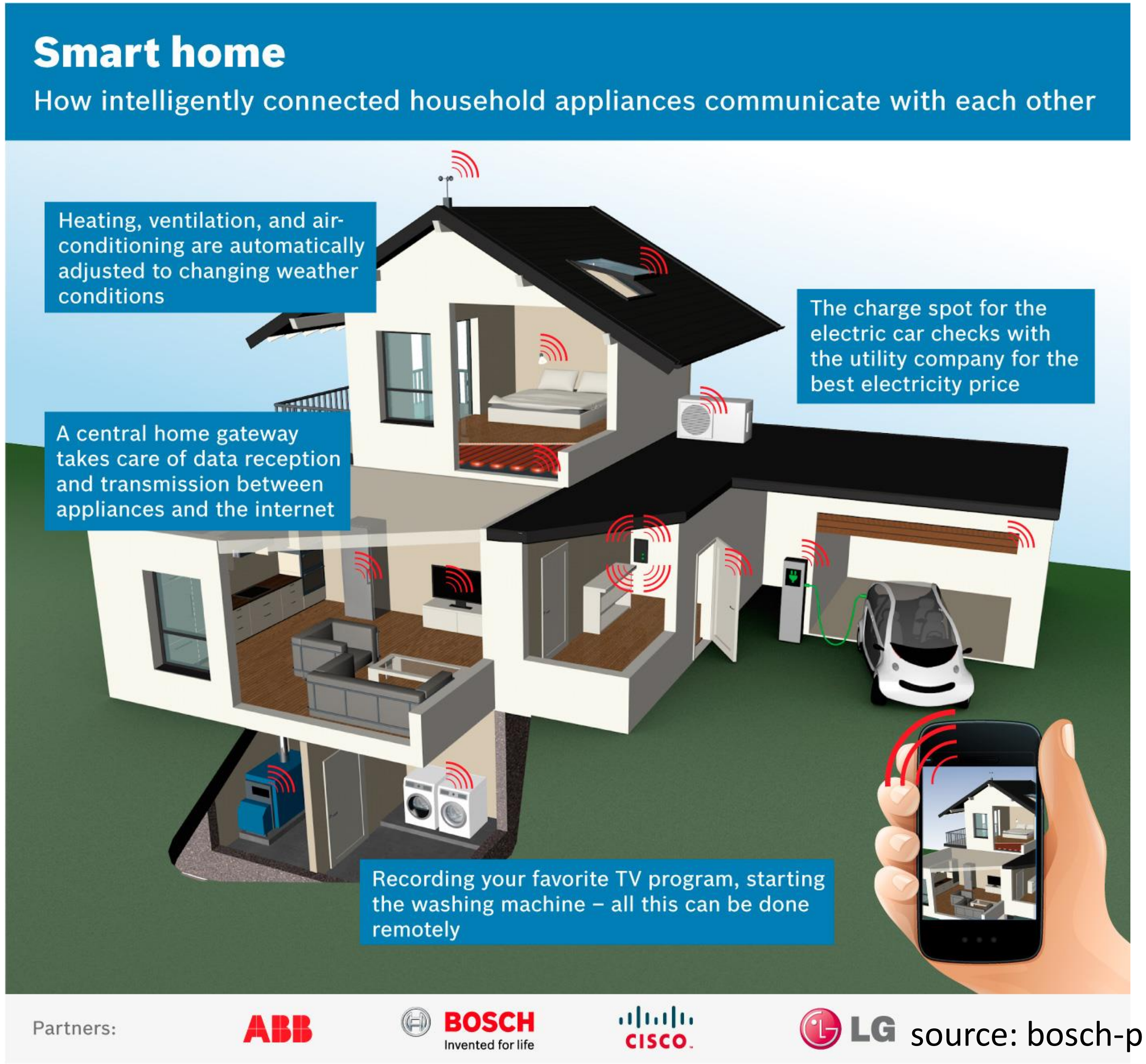
Facility **opening cost f_i**, for each i from I

- **Maximal Covering Location Problem** (MCLP)
  - Choose a subset of facilities to open so as to maximize the covered demand, **without exceeding a budget B** for opening facilities

- **Partial Set Covering Location Problem** (PSCLP)
  - Minimize the cost of open facilities that can **cover a certain fraction of the total demand**

A (not so) futuristic scenario

According to Gartner, a typical family home could contain more than 500 smart devices by 2022[1].



Smart home

How intelligently connected household appliances communicate with each other

Heating, ventilation, and air-conditioning are automatically adjusted to changing weather conditions

The charge spot for the electric car checks with the utility company for the best electricity price

A central home gateway takes care of data reception and transmission between appliances and the internet

Recording your favorite TV program, starting the washing machine – all this can be done remotely

source: bosch-presse.de

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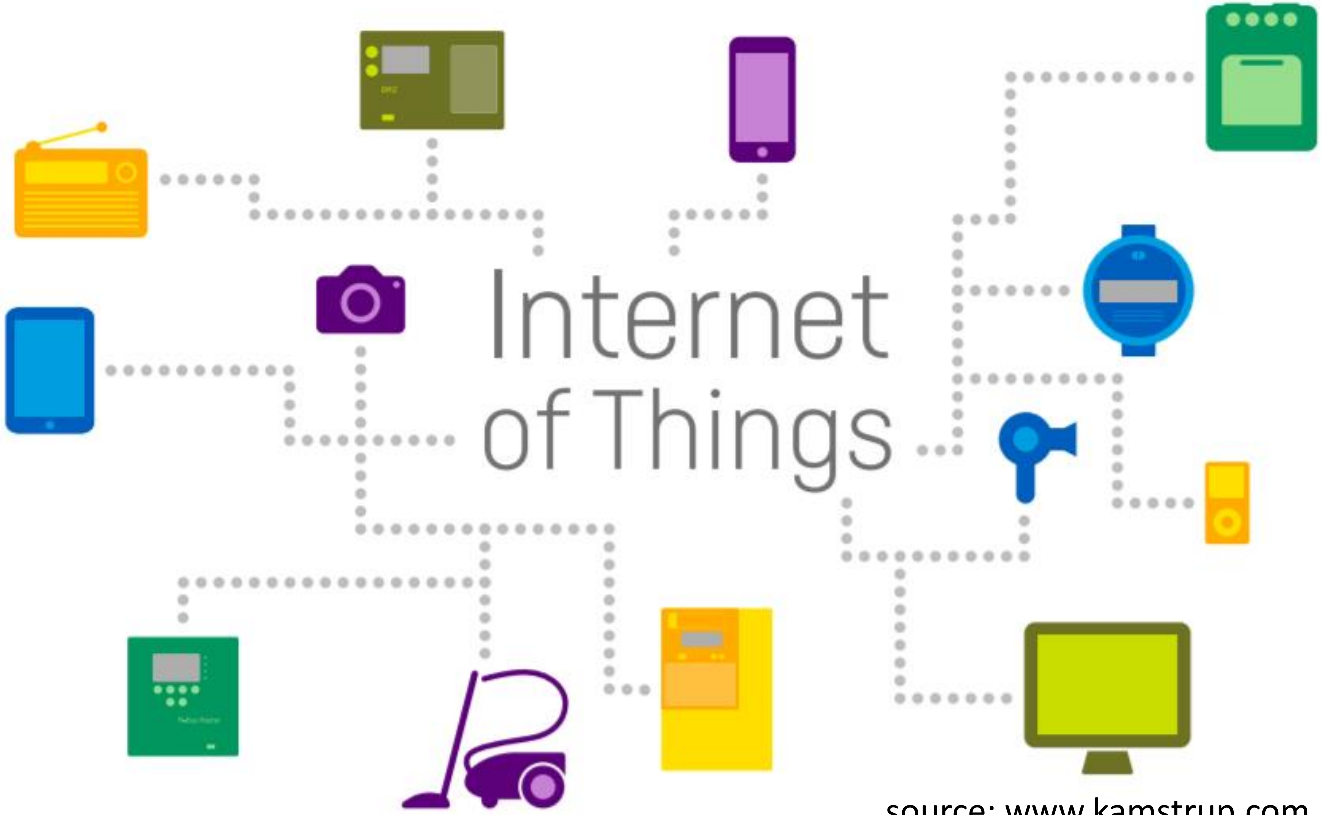
[1] (http://www.gartner.com/newsroom/ id/2839717)

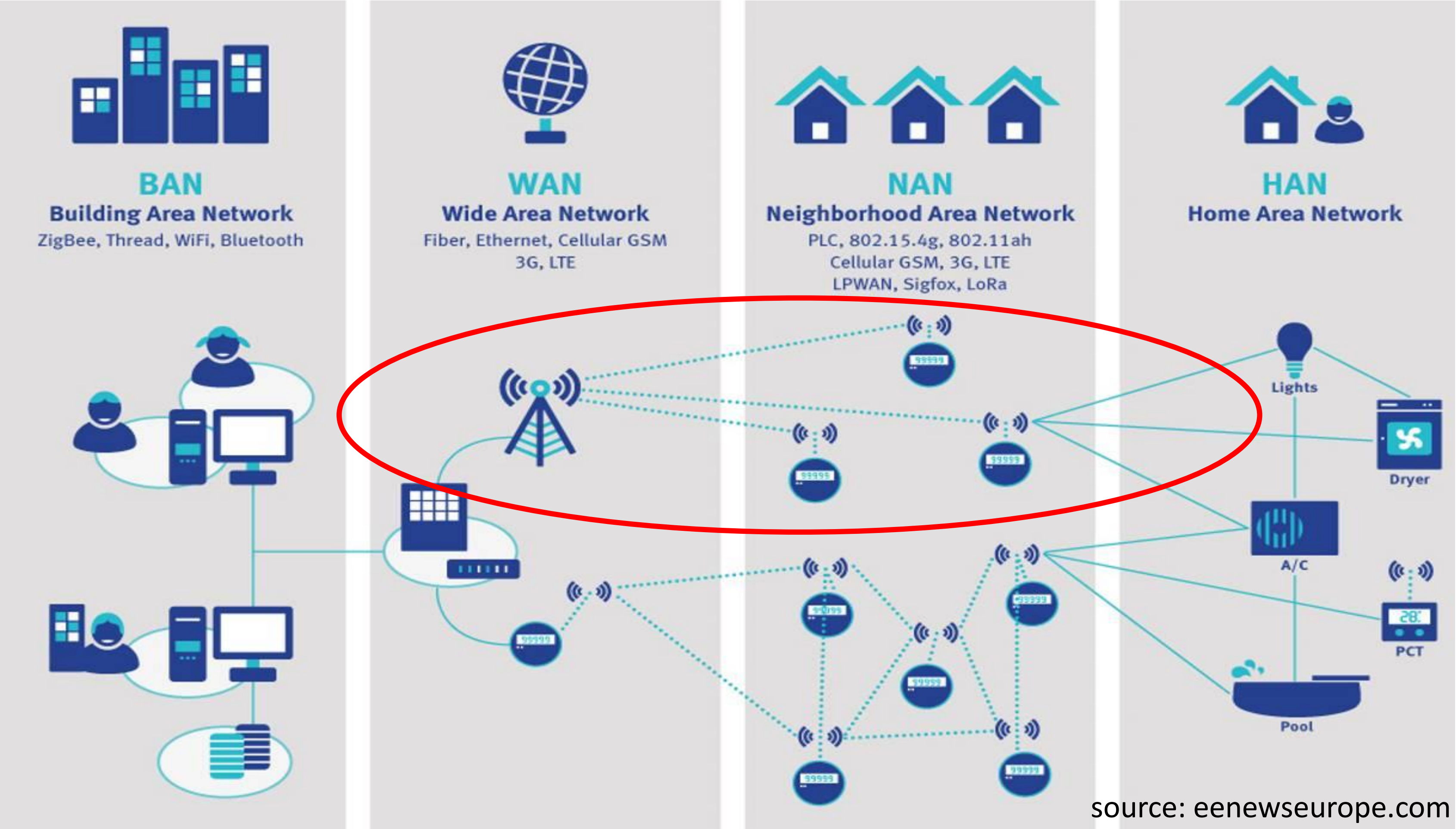
Smart Metering: beyond the simple billing function

- IoT: even disposable objects, such as milk cartons, will be perceptible in the digital world soon
- **Smart metering** is a driving force in making IoT a reality
- To interact with our surroundings through **data mining** and detailed **analytics**:
    - limiting energy consumption,
    - preserving resources
    - having e-devices operate according to our preferences
- Economic and environmental benefits



source: www.kamstrup.com

Wireless Communication



source: eenewseurope.com

(1) Point-to-Point, (2) Mesh Topology or (3) Hybrid

Smart Metering: Facility Location with BigData

- Given a set of households (with smart meters), decide where to place the **collection points/base stations for point-to-point communication** so as to:
  - Maximize the number of covered households given a certain budget for investing in the infrastructure → **MCLP**
  - Minimize the investment budget for covering a certain fraction of all households → **PSCLP**

Other Applications

- Service Sector:
  - Hospitals, libraries, restaurants, retail outlets

- Location of emergency facilities or vehicles:
  - fire stations, ambulances, oil spill equipments

- Continuous location covering (after discretization)

Related Literature

- **MCLP**, **heuristics**:
  - Church and ReVelle, 1974 (greedy heuristic)
  - Galvao and ReVelle, EJOR, 1996 Lagrangean heuristic
  - …
  - Maximo et al., COR, 2017
- **MCLP**, **exact methods**:
  - Downs and Camm, NRL, 1994 (branch-and-bound, Lagrangian relaxation)
- **PSCLP**:
  - Daskin and Owen, 1999, Lagrangian heuristic

Our Contribution

- Consider problems with very-large scale data
- Number of **demand points** runs in **millions** (big data)
- Relatively **low number** of potential facility locations

- We provide an exact solution approach for PSCLP and MCLP
- Based on **Branch-and-Benders-cut** approach
- The instances considered in this study are out of reach for modern MIP solvers

Benders Decomposition and Location Problems

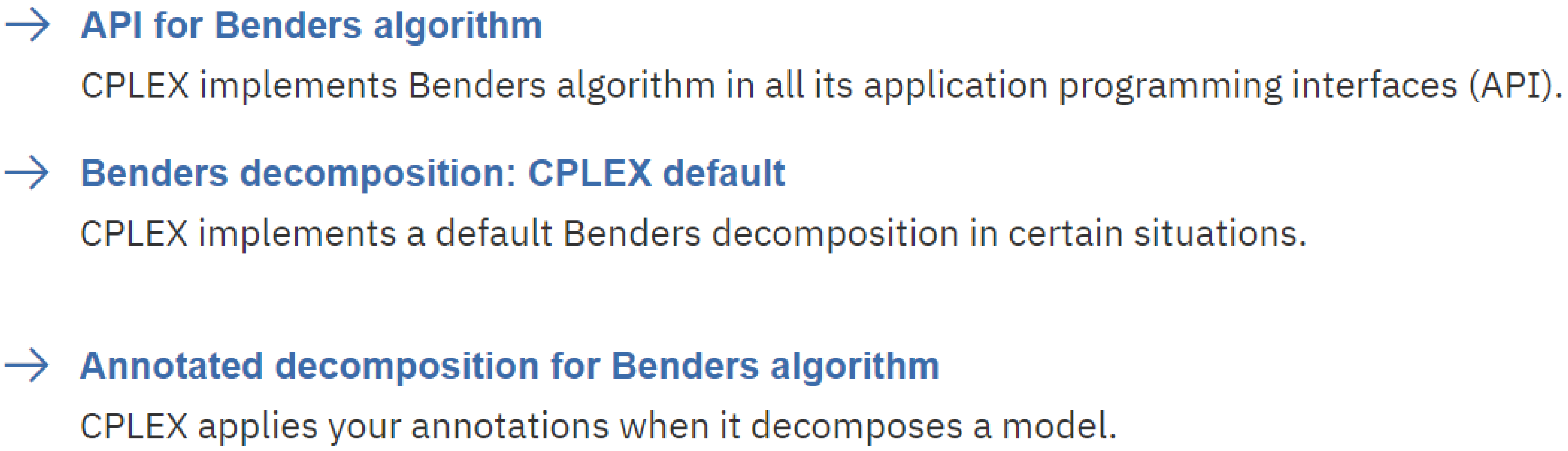
- With **sparse MILP formulations**, we can now solve to optimality:

  - Uncapacitated FLP (linear & quadratic)
    - (Fischetti, Ljubic, Sinnl, Man Sci 2017): 2K facilities x 10K clients
  - Capacitated FLP (linear & convex)
    - (Fischetti, Ljubic, Sinnl, EJOR 2016): 1K facilities x 1K clients
  - Maximum capture FLP with random utilities **(nonlinear)**
    - (Ljubic, Moreno, EJOR 2017): ~100 facilities x **80K clients**
  - Recoverable Robust FLP
    - (Alvarez-Miranda, Fernandez, Ljubic, TRB 2015): 500 nodes and 50 scenarios

- **Common to all: Branch-and-Benders-Cut**

Benders is trendy...

From CPLEX 12.7:



→ **API for Benders algorithm**

CPLEX implements Benders algorithm in all its application programming interfaces (API).

→ **Benders decomposition: CPLEX default**

CPLEX implements a default Benders decomposition in certain situations.

→ **Annotated decomposition for Benders algorithm**

CPLEX applies your annotations when it decomposes a model.

From SCIP 6.0

Compact MIP Formulations

The Partial Set Covering Location Problem

$$y_i = \begin{cases} 1, & \text{if facility } i \text{ is open,} \\ 0, & \text{otherwise} \end{cases} \qquad i \in I$$

$$z_j = \begin{cases} 1, & \text{if client } j \text{ is covered,} \\ 0, & \text{otherwise} \end{cases} \qquad j \in J$$

$$I_j = \sum_{i \in I(j)} y_i$$

$I(j)$: facilities that can cover client j

$$\begin{aligned} \min \quad & \sum_{i \in I} f_i y_i & \\ & \sum_{i \in I(j)} y_i \geq z_j & j \in J \\ & \sum_{j \in J} d_j z_j \geq D & \\ & y_i \in \{0,1\} & i \in I \\ & z_j \in \{0,1\} & j \in J. \end{aligned}$$

The Maximum Covering Location Problem

$$y_i = \begin{cases} 1, & \text{if facility } i \text{ is open,} \\ 0, & \text{otherwise} \end{cases} \qquad i \in I$$

$$z_j = \begin{cases} 1, & \text{if client } j \text{ is covered,} \\ 0, & \text{otherwise} \end{cases} \qquad j \in J$$

$$I_j = \sum_{i \in I(j)} y_i$$

$I(j)$: facilities that can cover client j

$$\max \quad \sum_{j \in J} d_j z_j$$

$$\sum_{i \in I(j)} y_i \geq z_j \qquad j \in J$$

$$\sum_{i \in I} f_i y_i \leq B$$

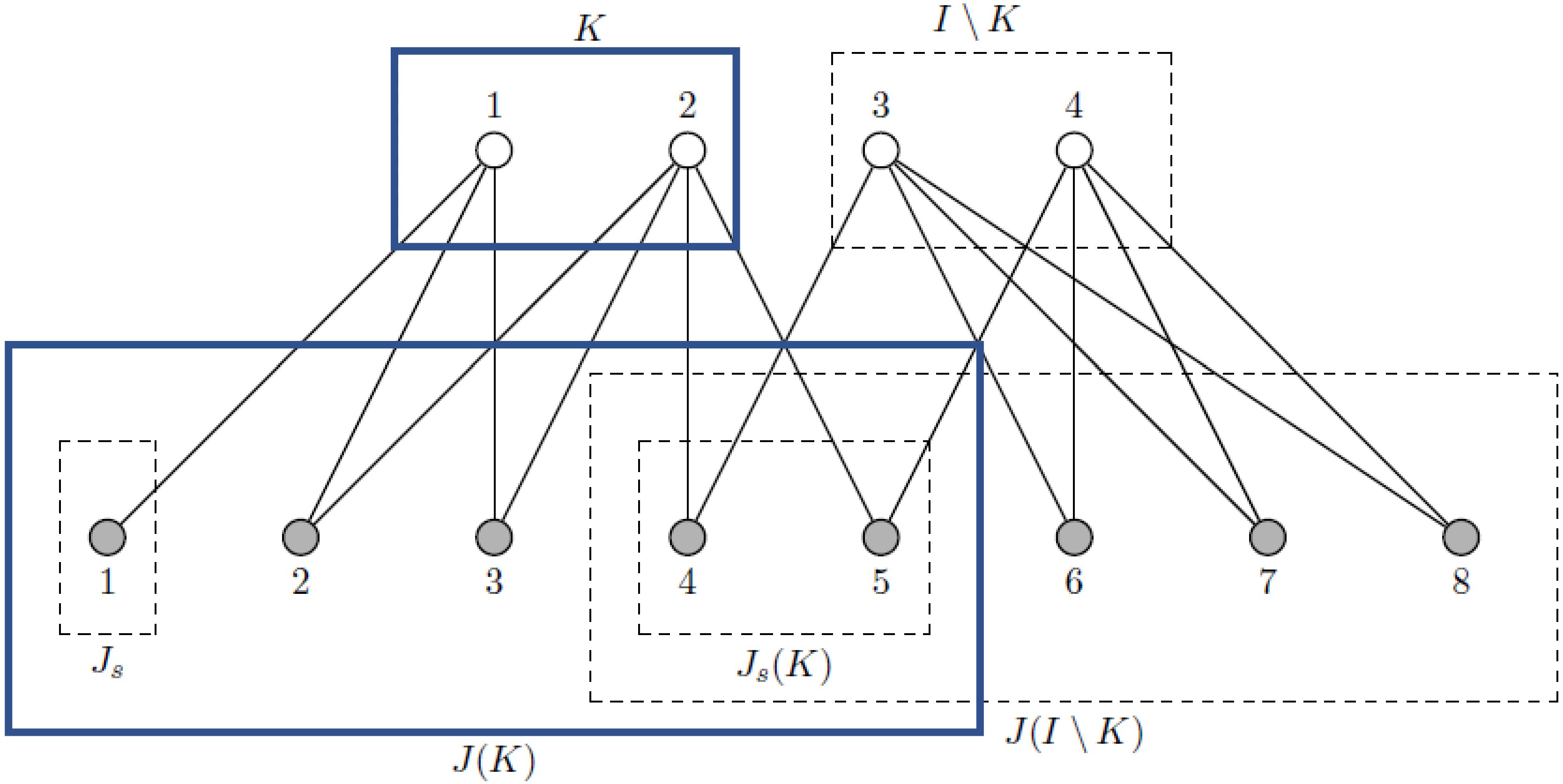
$$y_i \in \{0,1\} \qquad i \in I$$

$$\xcancel{z_j \in \{0,1\}} \qquad j \in J.$$

$$z_j \leq 1$$

Property: Integrality on z variables can be relaxed into $z_j \leq 1$, $j \in J$.

Notation



Benders Decomposition

For the PSCLP

Textbook Benders for the PSCLP

Master: $\min\left\{\sum_{i\in I} f_i y_i : \boxed{B_t(y) \geq 0,\ t \in P,}\ y_i \in \{0,1\},\ i \in I\right\},$

Branch-and-Benders-cut

For a given vector $\tilde{y} \in [0,1]^{|I|}$, the Benders primal subproblem is:

$$\min\left\{0 : z_j \leq \tilde{I}_j,\ j \in J,\quad \sum_{j\in J} d_j z_j \geq D,\quad 0 \leq z_j \leq 1,\ j \in J\right\}.$$

Separation:
Solve (1), if unbounded,
generate Benders cut

Its dual is given as:

$$\max\left\{D\gamma - \sum_{j\in J}\left(\tilde{I}_j \pi_j + \sigma_j\right) : (\pi,\sigma,\gamma) \in P\right\},$$

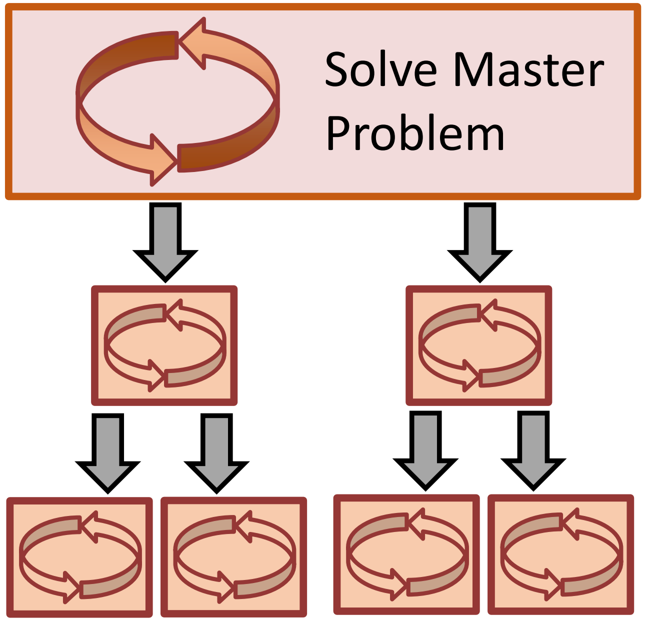
$$\sum_{i\in I}\left(\sum_{j\in J(i)} \tilde{\pi}_j\right) y_i \geq D\tilde{\gamma} - \sum_{j\in J} \tilde{\sigma}_j.$$

where π, γ and σ are dual variables constrained to belong to the pointed cone P:

$$P = \{(\pi,\gamma,\sigma) \geq 0 :\ \pi_j + \sigma_j \geq d_j\gamma,\ j \in J\}. \qquad (2)$$

A Careful Branch-and-Benders-Cut Design

- **Separation:** A fast algorithm for finding an optimal solution of the sub-problem.
- Which solution to choose?
- Stabilization techniques?
- A good **balance** between "lazy cut separation" (integer points only) and "user cut separation" (fractional points).



→ **Branch-and-Benders-Cut**

- **Crucial:** specialized procedures/combinatorial algorithms for the subproblem (if available).

Some Issues When Implementing Benders…

- Subproblem LP is **highly degenerate**, which Benders cut to choose?
- Pareto-optimal cuts, normalization, facet-defining cuts, etc
- MIP Solver may return a random (not necessarily extreme) ray of P
- The structure of P is quite simple – is there a better way to obtain an extreme ray of P (or extreme point of a normalized P)?

Normalization Approach

Observation: The solution $\tilde{y}$ of the restricted master problem is infeasible and only if the demand covered by $\tilde{y}$ is strictly less than D.

Branch-and-Benders-cut

Instead of solving a feasibility LP, we search for the maximum demand that can be covered by $\tilde{y}$:

$$\Delta(\tilde{y}) = \max \left\{ \sum_{j \in J} d_j z_j : z_j \le \tilde{I}_j,\ j \in J, \quad 0 \le z_j \le 1,\ j \in J \right\}.$$

Separation:
Solve Δ(y), if less than D,
generate Benders cut

This problem is always feasible, and its extreme point corresponds to an extreme ray of P.

Normalization is obtained by setting $\gamma = 1$.

$$\sum_{i \in I} \left(\sum_{j \in J(i)} \tilde{\pi}_j \right) y_i \ge D - \sum_{j \in J} \tilde{\sigma}_j.$$

Combinatorial Separation Algorithm: Cuts (B0) and (B0f)

Theorem: An optimal dual solution $(\tilde{\pi}, \tilde{\sigma})$ of the normalized Benders subproblem can be computed as:

$$\tilde{\pi}_j = \begin{cases} d_j, & \text{if } \tilde{I}_j < 1 \\ 0, & \text{otherwise} \end{cases} \qquad \tilde{\sigma}_j = \begin{cases} 0, & \text{if } \tilde{I}_j < 1 \\ d_j, & \text{otherwise} \end{cases} \qquad j \in J.$$

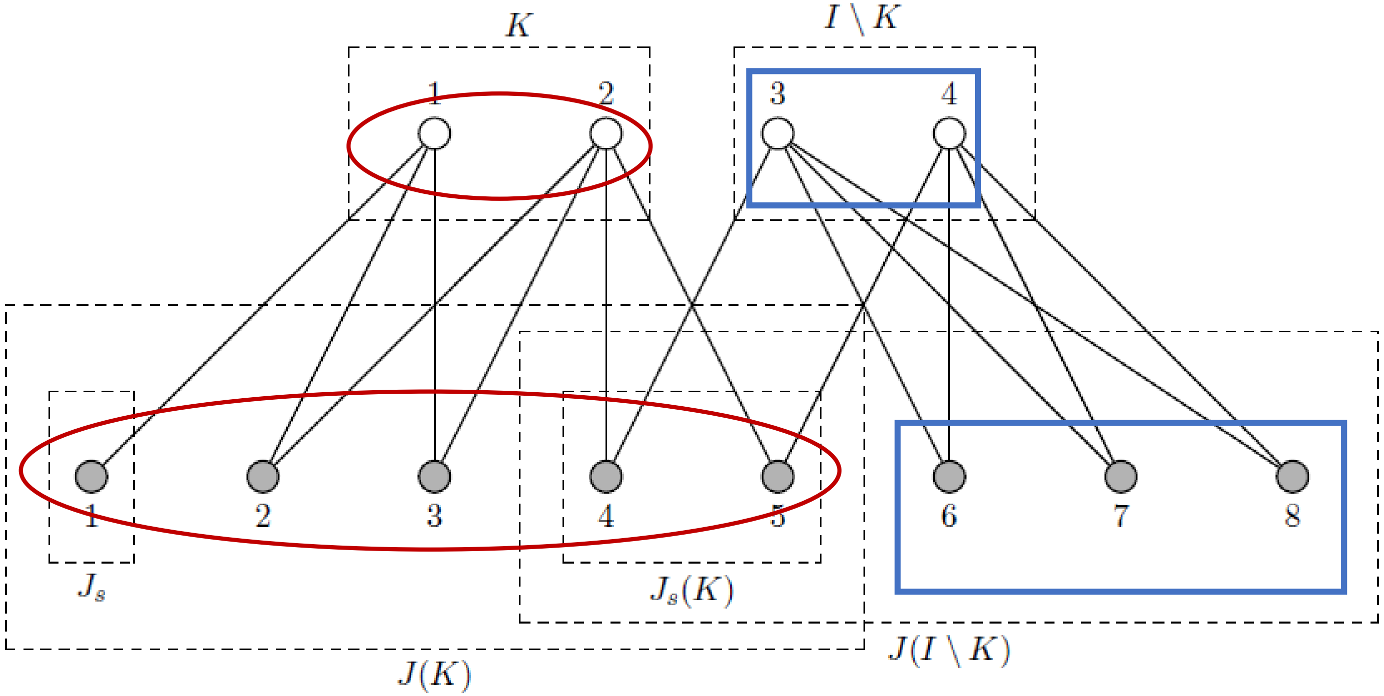
(B0f)

$$\sum_{i \in I} \left(\sum_{j \in J(i)} \tilde{\pi}_j \right) y_i \geq D - \sum_{j \in J} \tilde{\sigma}_j.$$

For a given point y, these cuts can be separated in linear time!

Benders Cuts (B0), derived at an integer point:

$$\sum_{i \notin \tilde{K}} \left(\sum_{j \in J(i) \setminus J(\tilde{K})} d_j \right) y_i \geq \underbrace{D - D(J(\tilde{K}))}_{\text{residual demand}} \quad \tilde{K} \subset I \qquad \text{(B0)}$$



Benders Cuts (B0), derived at an integer point:

$$\boxed{\sum_{i \notin \tilde{K}} \left(\sum_{j \in J(i) \setminus J(\tilde{K})} d_j \right) y_i} \geq \underbrace{D - D(J(\tilde{K}))}_{\text{residual demand}} \quad \tilde{K} \subset I \qquad \text{(B0)}$$

Combinatorial Separation Algorithm: Cuts (B1) and (B1f)

Theorem: An optimal dual solution $(\tilde{\pi}, \tilde{\sigma})$ of the normalized Benders subproblem can be computed as:

$$\tilde{\pi}_j = \begin{cases} d_j, & \text{if } \tilde{I}_j < 1 \text{ or } j \in J_s \\ 0, & \text{otherwise} \end{cases} \qquad \tilde{\sigma}_j = \begin{cases} 0, & \text{if } \tilde{I}_j < 1 \text{ or } j \in J_s \\ d_j, & \text{otherwise} \end{cases} \qquad j \in J.$$

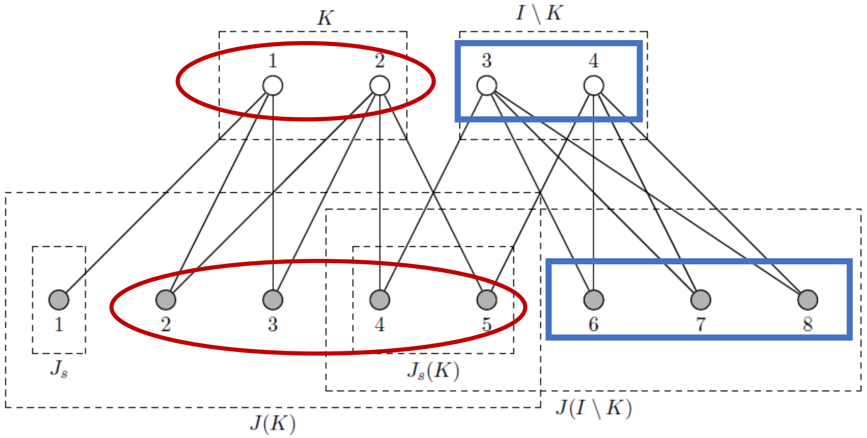
(B1f)

$$\boxed{\sum_{i \in I} \left(\sum_{j \in J(i)} \tilde{\pi}_j \right) y_i \geq D - \sum_{j \in J} \tilde{\sigma}_j.}$$

For a given point y, these cuts can be separated in linear time!

Benders Cuts (B1), derived at an integer point:

$$\boxed{\sum_{i \notin \tilde{K}} \left(\sum_{j \in J(i) \setminus J(\tilde{K})} d_j \right) y_i} + \sum_{i \in \tilde{K}} \left(\sum_{j \in J_s \cap J(i)} d_j \right) y_i \geq \underbrace{\boxed{D - D(J(\tilde{K}) \setminus J_s)}}_{\text{residual demand}} \qquad \tilde{K} \subset I$$



Combinatorial Separation Algorithm: Cuts (B2) and (B2f)

Theorem: An optimal dual solution $(\tilde{\pi}, \tilde{\sigma})$ of the normalized Benders subproblem can be computed as:

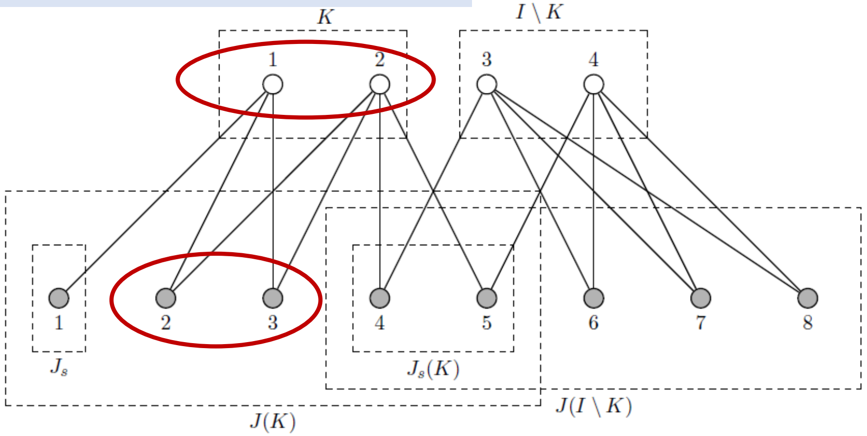
$$\tilde{\pi}_j = \begin{cases} d_j, & \text{if } \tilde{I}_j \leq 1 \\ 0, & \text{otherwise} \end{cases} \qquad \tilde{\sigma}_j = \begin{cases} 0, & \text{if } \tilde{I}_j \leq 1 \\ d_j, & \text{otherwise} \end{cases} \qquad j \in J.$$

(B2f)

$$\sum_{i \in I} \left(\sum_{j \in J(i)} \tilde{\pi}_j \right) y_i \geq D - \sum_{j \in J} \tilde{\sigma}_j.$$

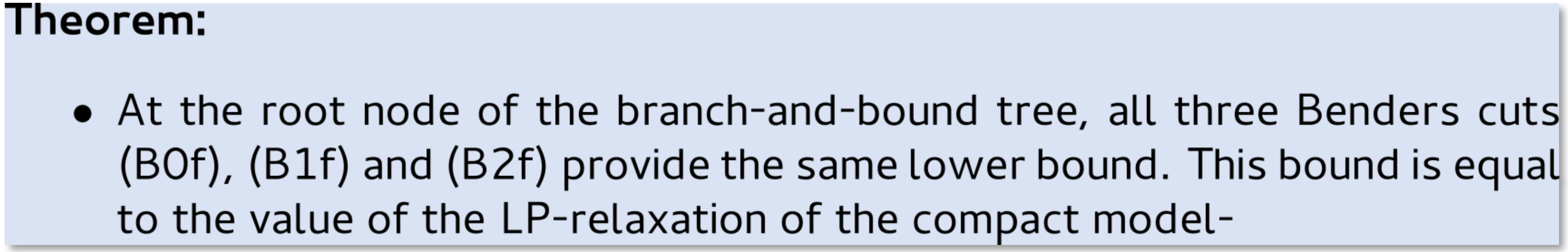
Benders Cuts (B2), derived at an integer point:

$$\sum_{i \notin \tilde{K}} \left(\sum_{j \in J(i) \setminus J(\tilde{K})} d_j + \sum_{j \in J(i) \cap J_s(\tilde{K})} d_j \right) y_i + \sum_{i \in \tilde{K}} \left(\sum_{j \in J_s(\tilde{K}) \cap J(i)} d_j \right) y_i \geq D - D(J(\tilde{K}) \setminus J_s(\tilde{K})) \quad \tilde{K} \subset I$$



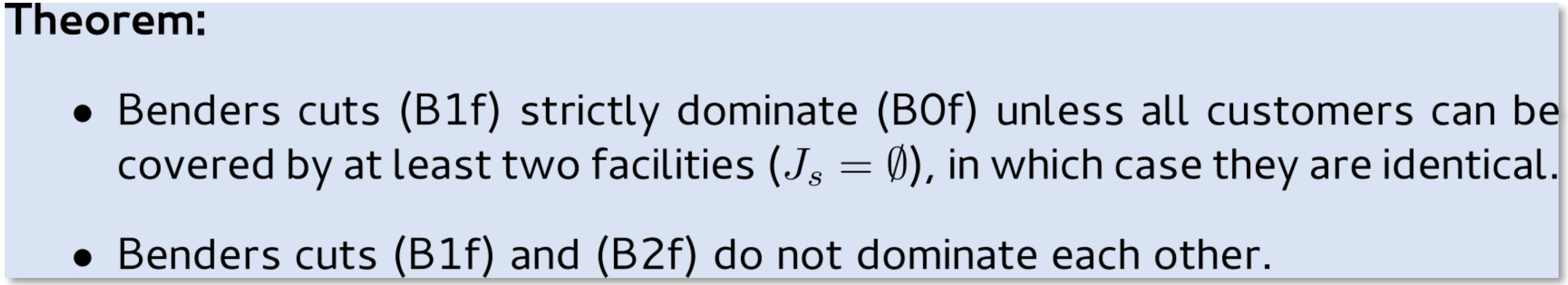
Comparing the Strength of Benders Cuts

Theorem:



- At the root node of the branch-and-bound tree, all three Benders cuts (B0f), (B1f) and (B2f) provide the same lower bound. This bound is equal to the value of the LP-relaxation of the compact model-

Theorem:



- Benders cuts (B1f) strictly dominate (B0f) unless all customers can be covered by at least two facilities ($J_s = \emptyset$), in which case they are identical.
- Benders cuts (B1f) and (B2f) do not dominate each other.

Facet-Defining Benders Cuts

- Following Conforti and Wolsey, 2016, one can use a CGLP to create a *facet-defining Benders cut* (facet of the LP-relaxation polytope).
- It requires a core point y^0 and the current fractional point $\tilde{y}$.
- It turns out, the CGLP corresponds to another normalization: intersection of the pointed cone P with another hyperplane.

$$
\begin{array}{lll}
\max & D\gamma - \sum_{j \in J} \sigma_j - \sum_{j \in J} \tilde{I}_j \pi_j & \\
 & \pi_j + \sigma_j \geq d_j \gamma & j \in J \\
 & \boxed{\sum_{j \in J} \left(I_j^0 - \tilde{I}_j \right) \pi_j \leq 1} & \\
 & (\pi, \sigma, \gamma) \geq 0. &
\end{array}
$$

What About MCLP?

Replace D by Theta in (B0f), (B1f), (B2f)

Master problem:

$$\max\left\{\theta:\ \sum_{i\in I} f_i y_i \le B,\ \boxed{B'_t(y,\theta)\ge 0,\ t\in P',}\ y_i\in\{0,1\},\ i\in I\right\}$$

Subproblem:

$$\min\left\{\sum_{j\in J}\left(\tilde{I}_j\pi_j+\sigma_j\right):\ (\pi,\sigma)\in P'\right\}$$

where

$$P' = \left\{(\pi,\sigma) : \pi_j+\sigma_j \ge d_j,\ j\in J,\quad (\pi,\sigma)\ge 0\right\}.$$

P' is precisely P intersected with $\gamma = 1$.

Replace D by Theta in (B0), (B1), (B2)

Master problem:

$$\max\left\{\theta:\ \sum_{i\in I} f_i y_i \leq B,\ \ B_t'(y,\theta)\geq 0,\ t\in P',\ \ y_i\in\{0,1\},\ i\in I\right\}$$

Benders cuts for MCLP:

$$\sum_{i\in I}\left(\sum_{j\in J(i)}\tilde{\pi}_j\right) y_i \geq \theta - \sum_{j\in J}\tilde{\sigma}_j.$$

Benders cuts for PSCLP:

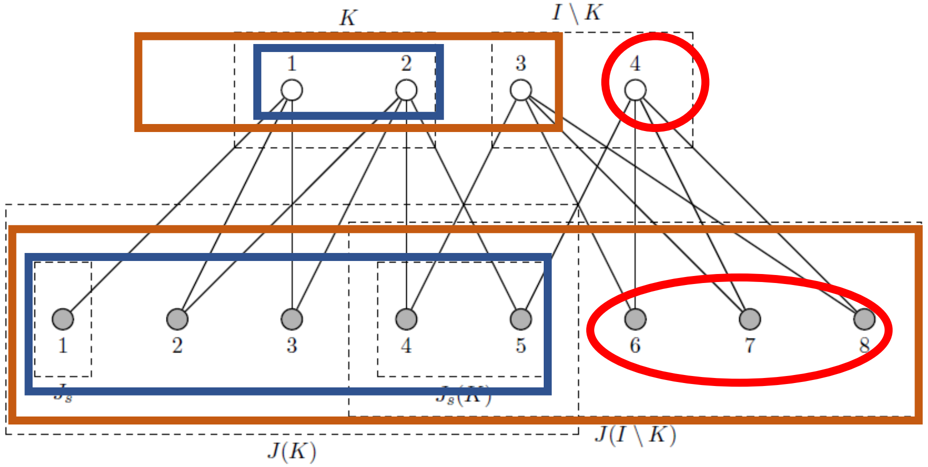
$$\sum_{i\in I}\left(\sum_{j\in J(i)}\tilde{\pi}_j\right) y_i \geq D - \sum_{j\in J}\tilde{\sigma}_j.$$

What About Submodularity?

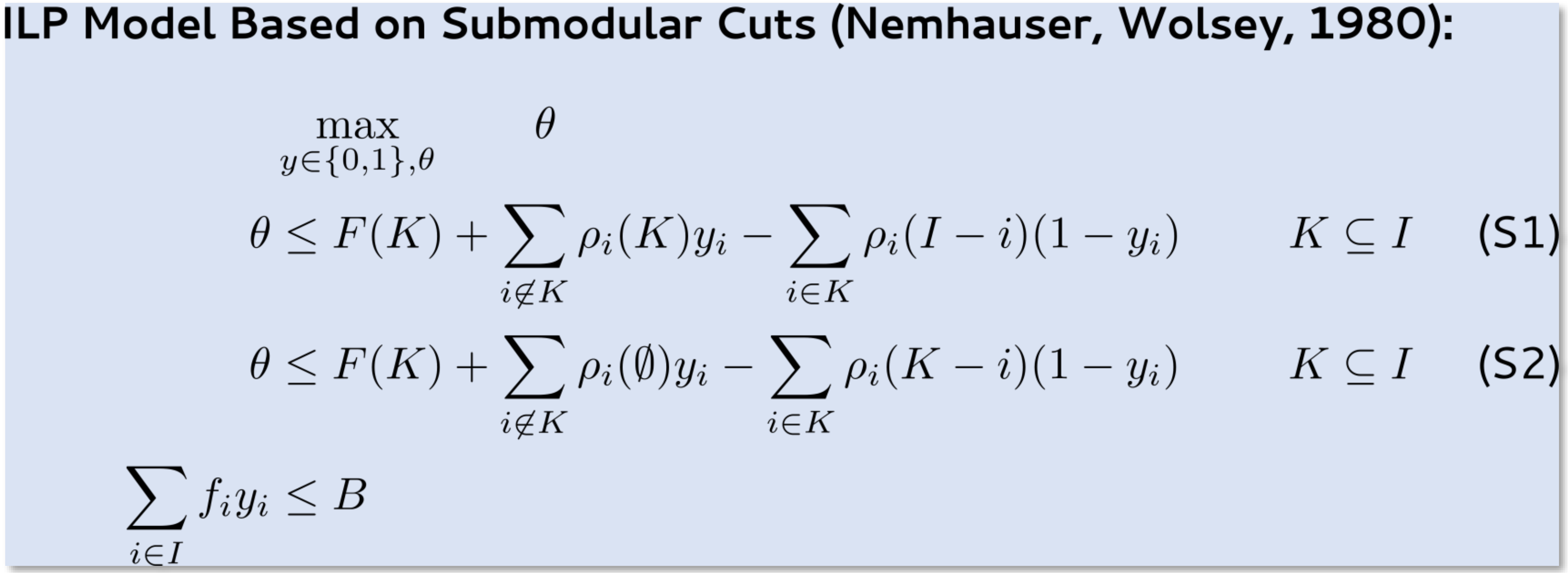
Benders Cuts vs Submodular Cuts

Covering Function $F(K)$ is submodular:

- For a given set of open facilities, $F(K)$ is the total demand covered by K.
- Let $\rho_i(K)$ be the **marginal contribution** of facility i, when the set K of facilities is already opened.
- A function is **submodular** if $\rho_i(K) \geq \rho_i(L)$, for all $K \subseteq L \subset I$, $i \notin L$.



Benders Cuts vs Submodular Cuts



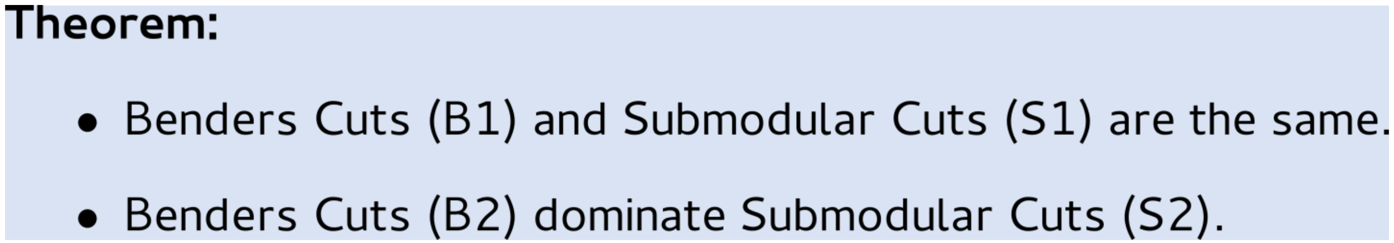
ILP Model Based on Submodular Cuts (Nemhauser, Wolsey, 1980):

$$\max_{y \in \{0,1\}, \theta} \quad \theta$$

$$\theta \leq F(K) + \sum_{i \notin K} \rho_i(K) y_i - \sum_{i \in K} \rho_i(I - i)(1 - y_i) \qquad K \subseteq I \qquad \text{(S1)}$$

$$\theta \leq F(K) + \sum_{i \notin K} \rho_i(\emptyset) y_i - \sum_{i \in K} \rho_i(K - i)(1 - y_i) \qquad K \subseteq I \qquad \text{(S2)}$$

$$\sum_{i \in I} f_i y_i \leq B$$



Theorem:

- Benders Cuts (B1) and Submodular Cuts (S1) are the same.
- Benders Cuts (B2) dominate Submodular Cuts (S2).

Computational Study

Benchmark Instances

- BDS (Benchmarking Data Set):
  - 10000, 50000, 100000 clients
  - 100 potential facilities
- MDS (Massive Data Set)
  - Between 0.5M and 20M clients

Budget	Covering Demand	Radius of Coverage
$B = 10$	$D = 50\%\bar{D}$	$\hat{R} \in \{5.5; 5.75; 6; 6.25\}$
$B = 15$	$D = 60\%\bar{D}$	$\hat{R} \in \{4; 4.25; 4.5; 4.75; 5\}$
$B = 20$	$D = 70\%\bar{D}$	$\hat{R} \in \{3.25; 3.5; 3.75; 4; 4.25\}$

Table 1: Random-coordinate data set parameters.

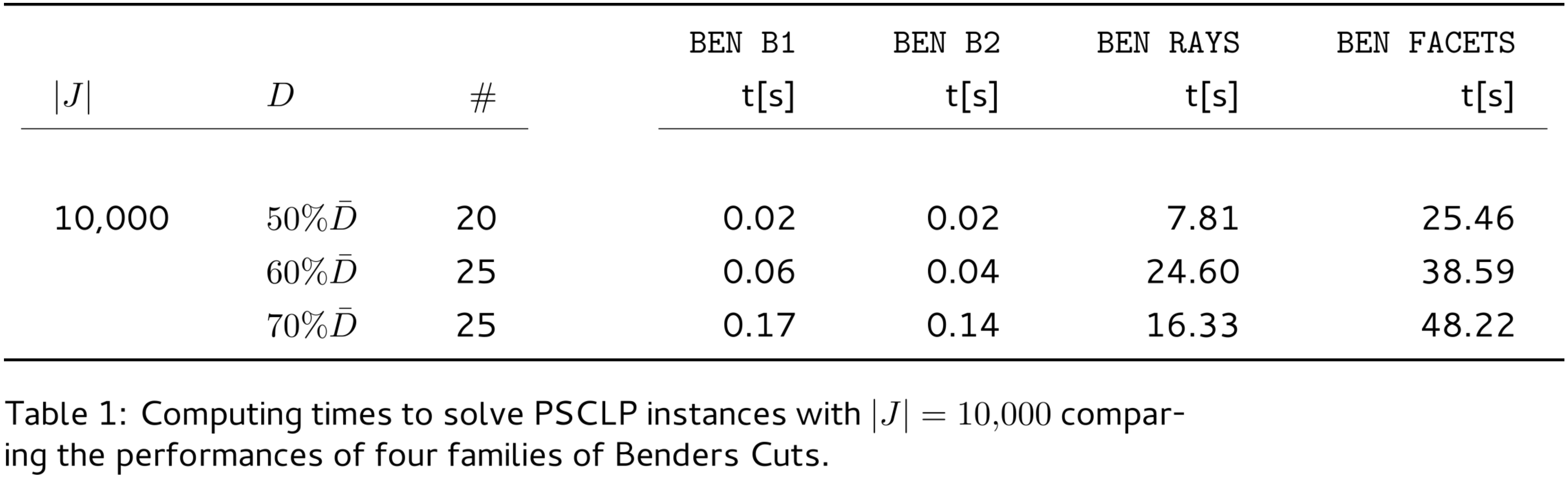
Tested Configurations

- `BEN B1`: where both fractional and integer points are separated using (B1f) and (B1) Benders cuts, respectively.
- `BEN B2`: where both fractional and integer points are separated using (B2f) and (B2) Benders cuts, respectively.
- `BEN RAYS`: where both fractional and integer points are separated using Benders cuts (B), whose coefficients are derived from extreme rays of the polyhedron associated to the dual LP of the Benders subproblem.
- `BEN FACETS`: where both fractional and integer points are separated using CGLP by Conforti and Wolsey.

In addition:

- `CPLEX`: compact model
- `AUTO BEN`: automatic Benders decomposition by Cplex

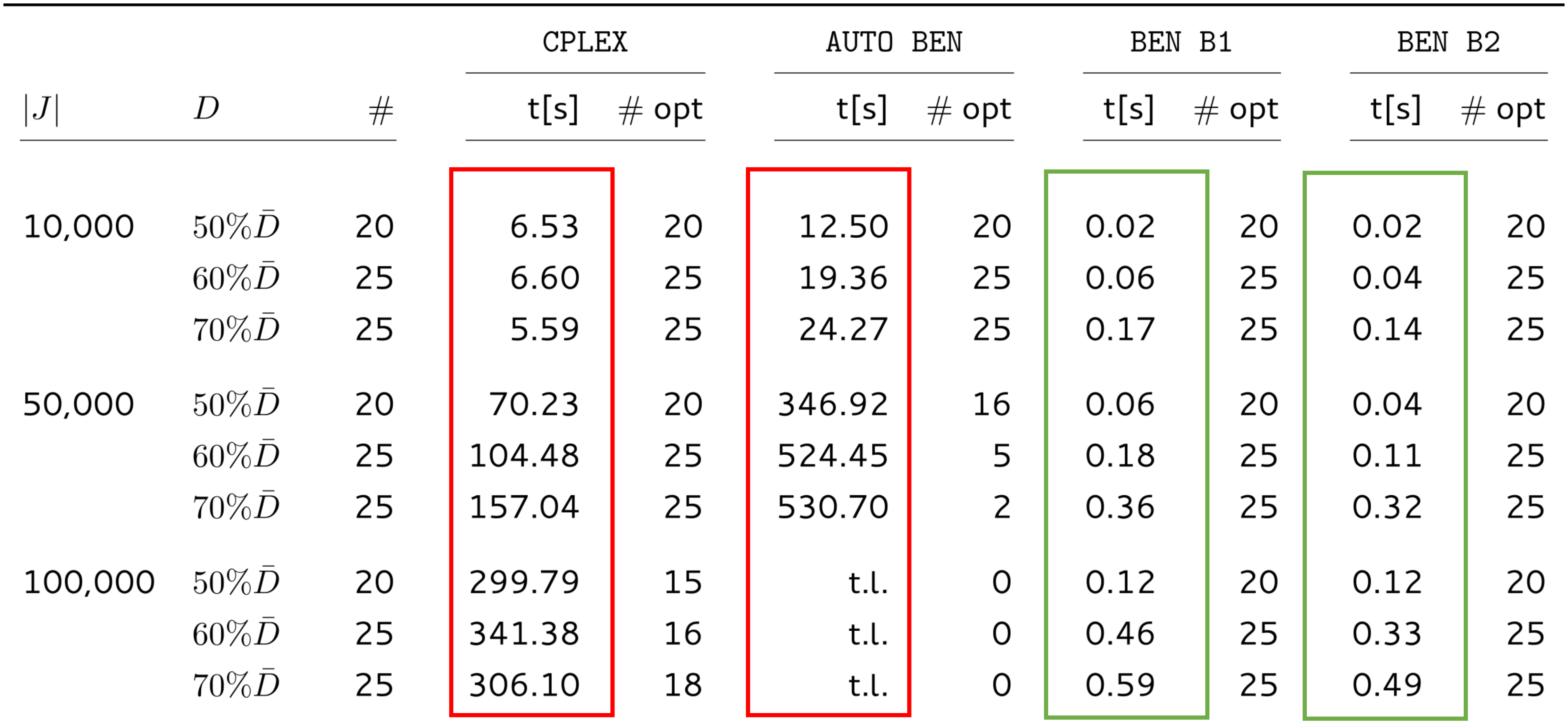
CPU Times for “Small” Instances



			BEN B1	BEN B2	BEN RAYS	BEN FACETS
$\lvert J\rvert$	D	#	t[s]	t[s]	t[s]	t[s]
10,000	$50\%\bar{D}$	20	0.02	0.02	7.81	25.46
	$60\%\bar{D}$	25	0.06	0.04	24.60	38.59
	$70\%\bar{D}$	25	0.17	0.14	16.33	48.22

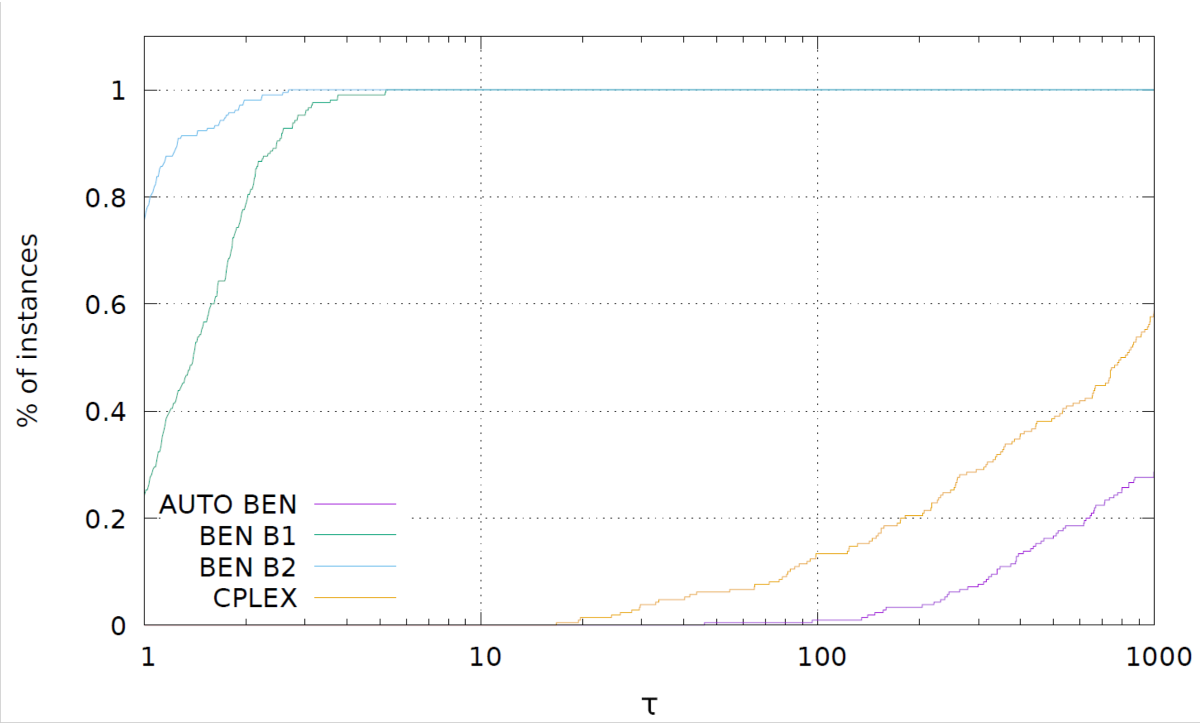
Table 1: Computing times to solve PSCLP instances with $|J| = 10{,}000$ comparing the performances of four families of Benders Cuts.

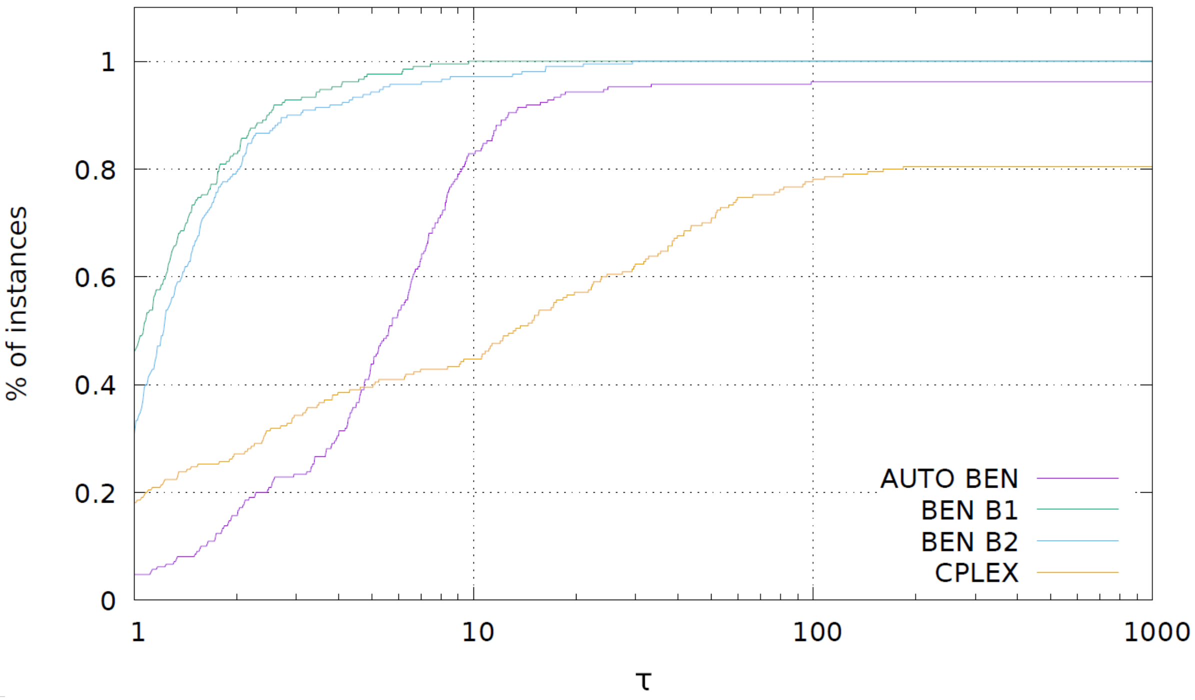
Comparison with CPLEX and Auto-Benders



			CPLEX		AUTO BEN		BEN B1		BEN B2	
$\|J\|$	D	#	t[s]	# opt	t[s]	# opt	t[s]	# opt	t[s]	# opt
10,000	$50\%\bar{D}$	20	6.53	20	12.50	20	0.02	20	0.02	20
	$60\%\bar{D}$	25	6.60	25	19.36	25	0.06	25	0.04	25
	$70\%\bar{D}$	25	5.59	25	24.27	25	0.17	25	0.14	25
50,000	$50\%\bar{D}$	20	70.23	20	346.92	16	0.06	20	0.04	20
	$60\%\bar{D}$	25	104.48	25	524.45	5	0.18	25	0.11	25
	$70\%\bar{D}$	25	157.04	25	530.70	2	0.36	25	0.32	25
100,000	$50\%\bar{D}$	20	299.79	15	t.l.	0	0.12	20	0.12	20
	$60\%\bar{D}$	25	341.38	16	t.l.	0	0.46	25	0.33	25
	$70\%\bar{D}$	25	306.10	18	t.l.	0	0.59	25	0.49	25

PSCLP vs MCLP





PSCLP on Instances with up to 20M clients

$\|J\|$	#	# opt	t[s]	# `BEN B2` int.	# `BEN B2` frac.	# nodes
500,000	70	70	1.75	4.20	40.00	64.90
1,000,000	70	70	4.14	3.77	31.16	61.03
1,500,000	70	70	7.51	4.20	31.36	58.49
2,000,000	70	70	8.51	4.33	28.94	44.76
4,000,000	70	70	23.35	3.80	31.51	64.04
6,000,000	70	70	28.20	3.99	30.17	48.51
10,000,000	70	70	55.76	3.80	34.96	60.50
15,000,000	70	70	109.61	4.39	42.44	80.47
20,000,000	70	70	117.25	5.12	36.21	58.01

Table 1: Computational performance of `BEN B2` on massive PSCLP data sets.

To summarize…

- Two important location problems that have not received much attention in the literature despite their theoretical and practical relevance.

- The first exact algorithm to effectively tackle realistic PSCLP and MCLP instances with millions of demand points.

- These instances are far beyond the reach of modern general-purpose MIP solvers.

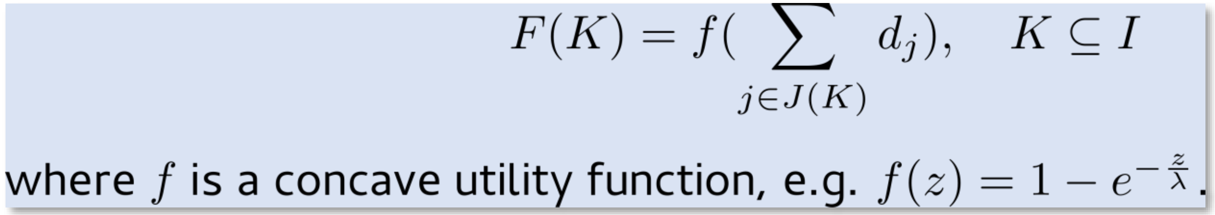
- Effective branch-and-Benders-cut algorithms exploits a combinatorial cut-separation procedure.

Interesting Directions for Future Work

- Problem variants under uncertainty (robust, stochastic)
- Multi-period, multiple coverage, facility location & network design
- Data-driven optimization
- Applications in clustering and classification

Exploiting **submodularity** together with **concave utility functions**

  - Benders Cuts
  - Outer Approximation
  - Submodular Cuts
  - In the original or in the projected space…

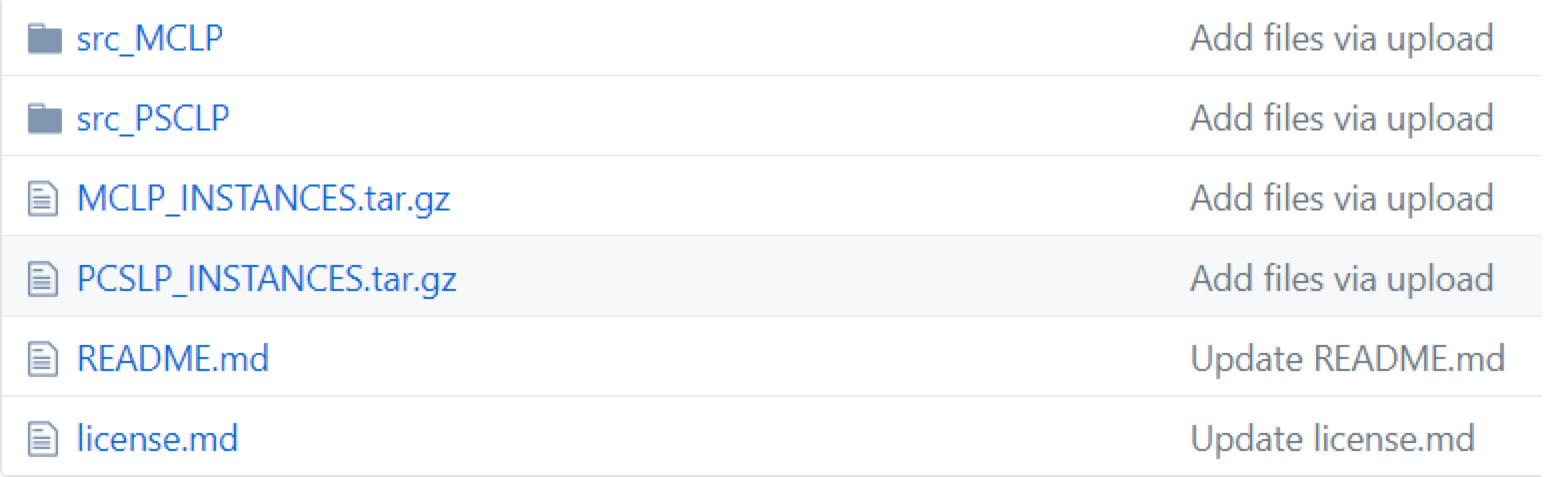


$$F(K) = f(\sum_{j\in J(K)} d_j), \quad K \subseteq I$$

where f is a concave utility function, e.g. $f(z) = 1 - e^{-\frac{z}{\lambda}}$.

Open-Source Implementation

https://github.com/fabiofurini/LocationCovering



src_MCLP	Add files via upload
src_PSCLP	Add files via upload
MCLP_INSTANCES.tar.gz	Add files via upload
PCSLP_INSTANCES.tar.gz	Add files via upload
README.md	Update README.md
license.md	Update license.md

J.F. Cordeau, F. Furini, I. Ljubic:
Benders Decomposition for Very Large Scale Partial Set Covering and Maximal Covering Problems,
European Journal of Operational Research, to appear, 2019