On Integer and Bilevel Formulations for the k-Vertex Cut Problem

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joint work with Fabio Furini[°], Enrico Malaguti[*] and Paolo Paronuzzi[*]

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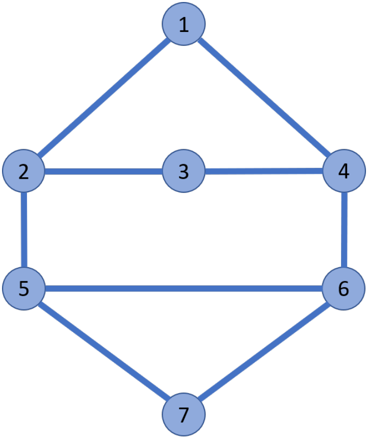
Problem setting and motivation

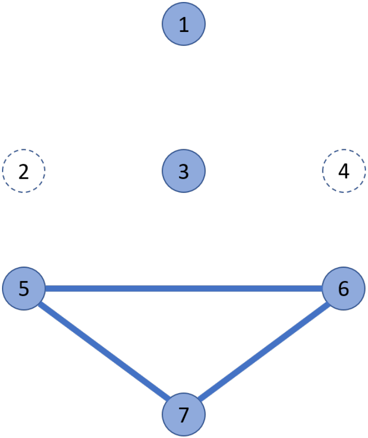
Problem setting

A **k-vertex cut** is a subset of vertices whose removal disconnects the graph in at least k (not-empty) components.



Problem setting

Example of a 3-vertex cut:



The k-Vertex Cut Problem

Definition

Given an undirected graph $G = (V, E)$ with vertex weights w_v, $v \in V$, and a integer $k \geq 2$, find a subset of vertices of **minimum weight** whose removal disconnects G in at least k (not-empty) components.

Motivation

- **Family of Critical Node Detection Problems** (M. Lalou, M. A. Tahraoui, and H. Kheddouci. The critical node detection problem in networks: A survey. Computer Science Review, 2018);
- **Analysis of networks** (D. Kempe, J. Kleinberg, and É. Tardos. Influential nodes in a diffusion model for social networks. Automata, Languages and Programming, 2005.);
- **Decomposition method for linear equation systems**.

Compact Model and Representative Formulation

Compact formulation

We associate a binary variable y_v^i to all vertices $v \in V$ and for all integers $i \in K$, such that:

$$y_v^i = \begin{cases} 1 & \text{if vertex } v \text{ belongs to component } i \\ 0 & \text{otherwise} \end{cases} \qquad i \in K, v \in V.$$

Compact Model

Compact ILP formulation for k-Vertex-Cut Problem:

$$\min \sum_{v \in V} w_v - \sum_{i \in K} \sum_{v \in V} w_v y_v^i$$

$$\sum_{i \in K} y_v^i \leq 1 \qquad v \in V,$$

$$y_u^i + y_v^j \leq 1 \qquad i \neq j \in K, uv \in E,$$

$$\sum_{v \in V} y_v^i \geq 1 \qquad i \in K,$$

$$y_v^i \in \{0, 1\} \qquad i \in K, v \in V.$$

Drawbacks: LP-optimal solution is zero (set all $y_v^i = 1/k$), symmetries, etc.

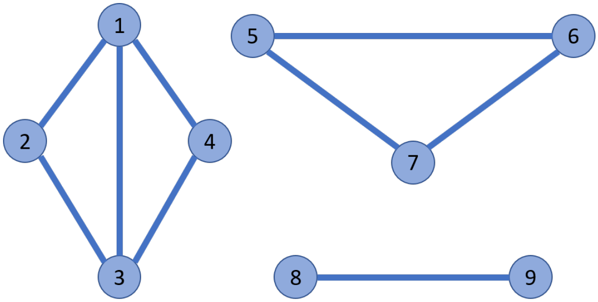
Bilevel approach

Bilevel approach

Property

A graph G has at least k (not empty) components if and only if any cycle-free subgraph of G contains at most $|V| - k$ edges.

Example with $|V| = 9$ and $k = 3$:



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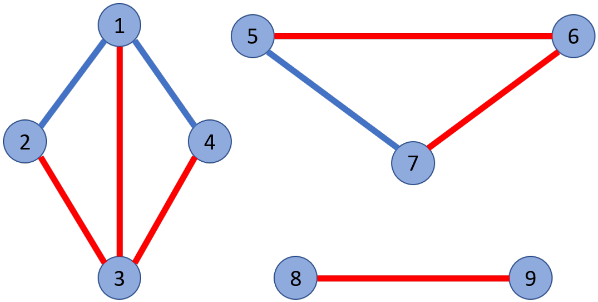


Bilevel approach

The k-vertex cut problem can be seen as a Stackelberg game:

- the leader searches the smallest subset of vertices V_0 to delete;
- the follower **maximizes the size of the cycle-free subgraph** on the residual graph.

Property

The solution $V_0 \subset V$ of the leader is feasible if and only if the value of the **optimal follower's response** *(i.e., the size of the maximum cycle-free subgraph in the remaining graph)* **is at most $|V| - |V_0| - k$.**

Bilevel approach

The leader decisions:

$$x_v = \begin{cases} 1 & \text{if vertex } v \text{ is in the } k\text{-vertex cut} \\ 0 & \text{otherwise} \end{cases} \qquad v \in V$$

For the decisions of the follower, we use additional binary variables associated with the edges of G:

$$e_{uv} = \begin{cases} 1 & \text{if edge } uv \text{ is selected to be in the cycle-free subgraph} \\ 0 & \text{otherwise} \end{cases} \qquad uv \in E$$

Bilevel approach

The Bilevel ILP formulation of the k-vertex cut problem reads as follows:

$$\min \sum_{v \in V} x_v$$

$$\Phi(x) \leq |V| - \sum_{v \in V} x_v - k$$

$$x_v \in \{0, 1\} \qquad v \in V.$$

- $\Phi(x)$ is the optimal solution value of the follower subproblem for a given x.
- **Value Function Reformulation**.
- Value function $\Phi(x)$ is neither convex, nor concave, nor connected...

How do we calculate $\Phi(x)$?

For a solution x^* of the leader, which denotes a set V_0 of interdicted vertices, the follower's subproblem is:

$$\Phi(x^*) = \max \sum_{uv \in E} e_{uv}$$

$$\begin{aligned} e(S) &\leq |S| - 1 && S \subseteq V, S \neq \emptyset, \\ e_{uv} &\leq 1 - x^*_u && uv \in E, \\ e_{uv} &\leq 1 - x^*_v && uv \in E, \\ e_{uv} &\in \{0, 1\} && uv \in E. \end{aligned}$$

Bilevel approach

We can prove that the follower's subproblem is equivalently restated as:

$$\Phi(x^*) = \quad \max \sum_{uv \in E} z_{uv}(1 - x_u^* - x_v^*)$$

$$z(S) \leq |S| - 1 \qquad S \subseteq V, S \neq \emptyset$$

$$z_{uv} \in \{0, 1\} \qquad uv \in E.$$

- **Convexification** of the value function $\Phi(x)$

Bilevel approach

Since the space of feasible solutions of the redefined follower subproblem does not depend on the leader anymore, the non-linear constraint from the BILP formulation:

$$\Phi(x) \leq |V| - \sum_{v \in V} x_v - k$$

can now be replaced by the following exponential family of inequalities:

$$\sum_{uv \in E(T)} (1 - x_u - x_v) \leq |V| - \sum_{v \in V} x_v - k \qquad T \in \mathcal{T}$$

where $\mathcal{T}$ denote the **set of all cycle-free subgraphs** of G.

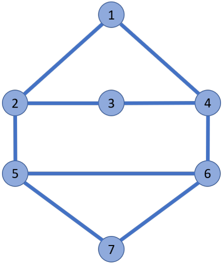
Natural Formulation

The following single-level formulation, denoted as *Natural Formulation*, is a valid model for the k-vertex cut problem:

$$
\begin{aligned}
\min \sum_{v \in V} w_v x_v & & \\
\sum_{v \in V} [\deg_T(v) - 1] x_v \geq k - |V| + |E(T)| & \qquad & T \in \mathcal{T}, \\
x_v \in \{0, 1\} & \qquad & v \in V.
\end{aligned}
$$

Natural formulation

$$\sum_{v \in V} [\deg_T(v) - 1] x_v \geq k - |V| + |E(T)|$$



Natural formulation

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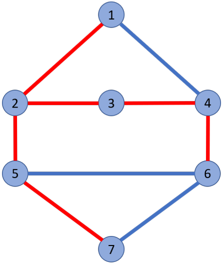

$$2x_2 + x_4 + x_5 \geq 2$$

Natural formulation

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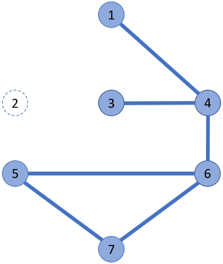

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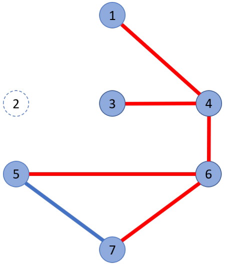
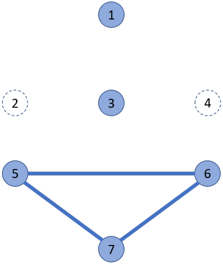


$$2x_2 + x_4 + x_5 \geq 2$$

$$-x_2 + 2x_4 + 2x_6 \geq 1$$

Natural formulation

$$\sum_{v \in V} [\deg_T(v) - 1] x_v \geq k - |V| + |E(T)|$$



$$2x_2 + x_4 + x_5 \geq 2$$

$$-x_2 + 2x_4 + 2x_6 \geq 1$$

Separation procedure

Let x^* be the current solution. We define edge-weights as

$$w^*_{uv} = 1 - x^*_u - x^*_v, \quad uv \in E$$

and search for the maximum-weighted cycle-free subgraph in G. Let W^* denote the weight of the obtained subgraph; if $W^* > |V| - k - \sum_{v \in V} x^*_v$, we have detected a violated inequality.

The separation procedure can be performed in polynomial time by running an adaptation of Kruskal's algorithm for minimum-spanning trees.

A Hybrid Approach

Representative Variables

Observation

- A graph G admits a k-vertex cut if and only if $\alpha(G) \geq k$.
- To each component we associate a vertex from the stable set - **a representative**.

We introduce a set of binary variable to select which vertices are representative:

$$z_v = \begin{cases} 1 & \text{if vertex } v \text{ is the representative of a component} \\ 0 & \text{otherwise} \end{cases} \qquad v \in V$$

Representative Constraints

$$\sum_{v \in V} z_v = k$$

$$z_u + z_v \leq 1 \qquad uv \in E,$$

$$x_u + z_u \leq 1 \qquad u \in V,$$

$$z_u + \sum_{v \in N(u)} z_v \leq 1 + (\deg(u) - 1)x_u \qquad u \in V.$$

Computational experiments

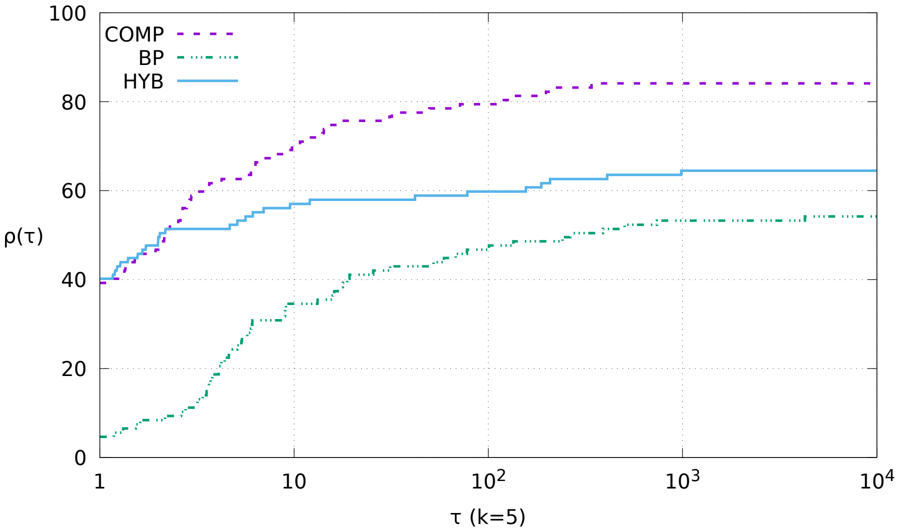
Computational Experiments

We considered two sets of graph instances from the 2nd DIMACS and 10th DIMACS challenges.
For all the instances we tested four different values of k (5, 10, 15, 20).

Compared Methods (time limit of 1 hour):

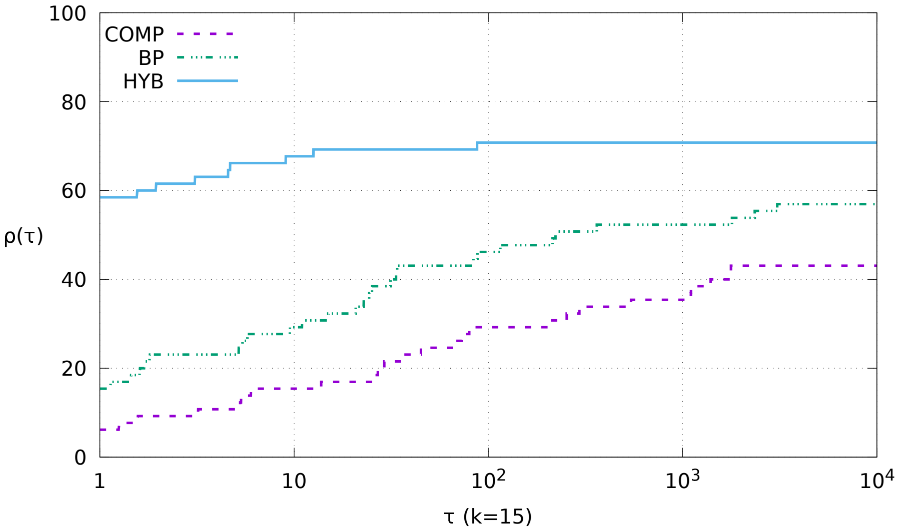
- **COMP:** *Compact* model (solved by CPLEX 12.7.1);
- **BP:** State-of-the-art Branch-and-Price solving an *Extended* formulation (*Cornaz, D., Furini, F., Lacroix, M., Malaguti, E., Mahjoub, A. R., & Martin, S. (2017). The Vertex k-cut Problem, Discrete Optimization, 2018.*);
- **HYB:** Hybrid approach

Case with $k = 5$

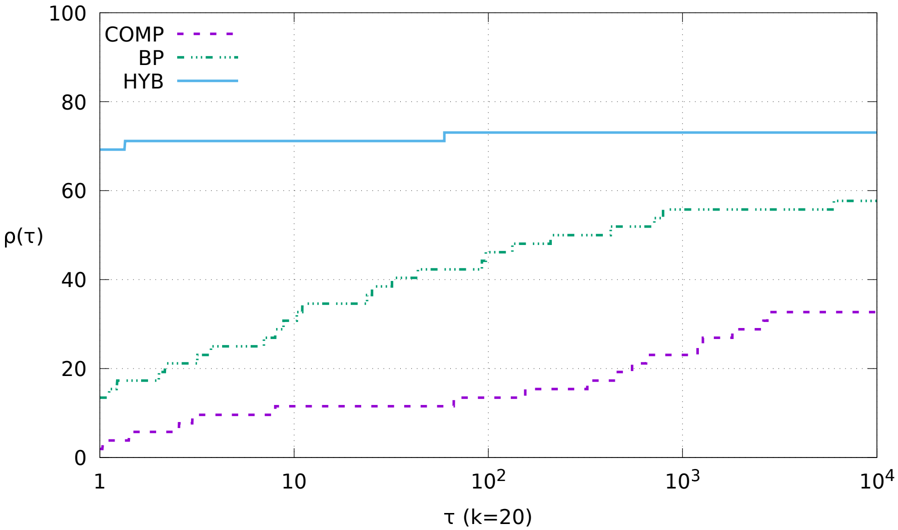


Computational Experiments

Case with $k = 15$



Case with $k = 20$



Conclusions and future work

- Our hybrid formulation outperforms both CPLEX and $B\&P$;
- It is a thin formulation, with $O(n)$ variables
- We partially exploit a **hereditary property** on G (if a subset of edges is cycle-free, any subset of it is cycle-free too) to convexify $\Phi(x)$
- This allows us to derive an ILP formulation in the natural space (was open for some time)
- Where else can we exploit similar ideas?

Conclusions and future work

Thank you for your attention.