FROM GAME THEORY TO GRAPH THEORY: A BILEVEL JOURNEY

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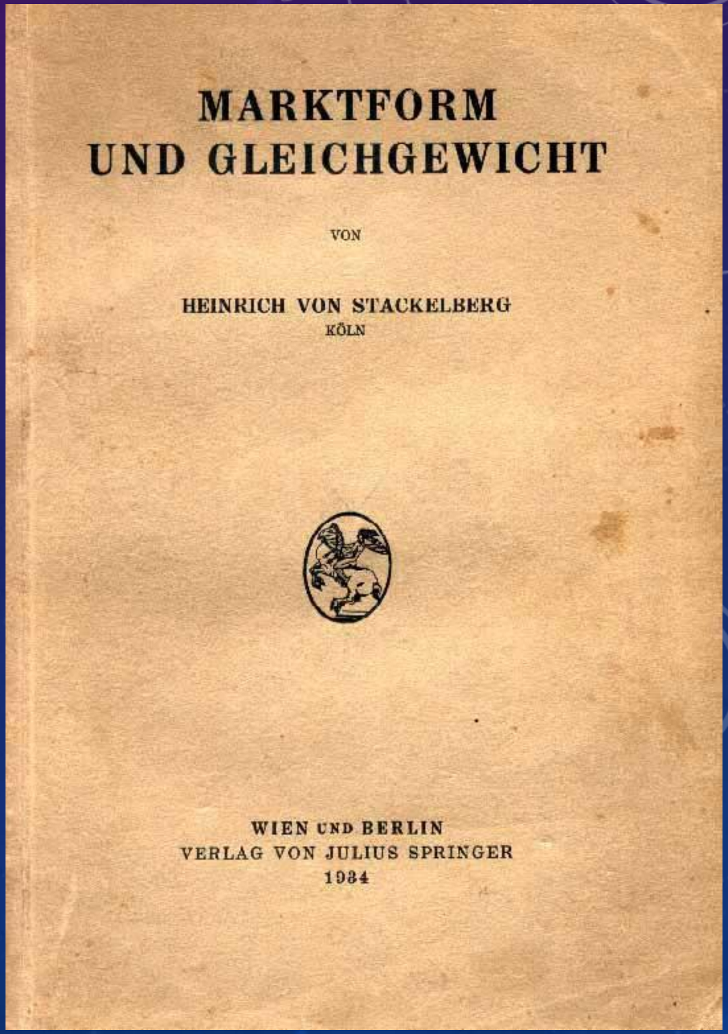
References:

- M. Fischetti, I. Ljubic, M. Monaci, M. Sinnl: Interdiction Games and Monotonicity, with Application to Knapsack Problems, *INFORMS Journal on Computing* 31(2):390-410, 2019
- F. Furini, I. Ljubic, P. San Segundo, S. Martin: The Maximum Clique Interdiction Game, *European Journal of Operational Research* 277(1):112-127, 2019
- F. Furini, I. Ljubic, E. Malaguti, P. Paronuzzi: On Integer and Bilevel Formulations for the k-Vertex Cut Problem, submitted, 2018
- M. Fischetti, I. Ljubic, M. Monaci, M. Sinnl: A new general-purpose algorithm for mixed-integer bilevel linear programs, *Operations Research* 65(6): 1615-1637, 2017

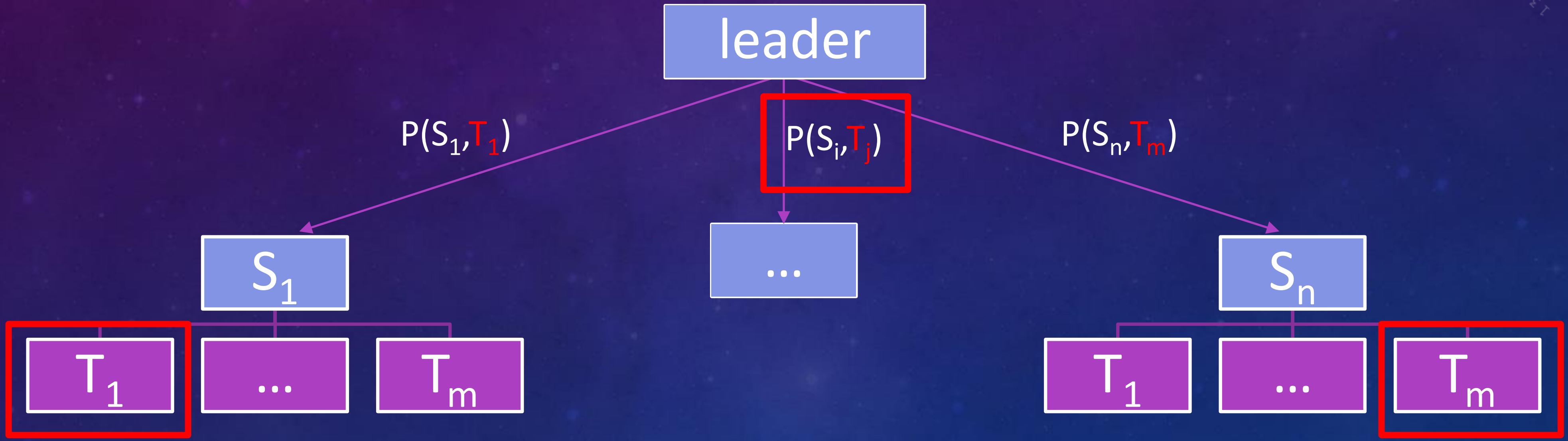
SOLVER: https://msinnl.github.io/pages/bilevel.html

STACKELBERG GAMES

- Introduced in economy by H. v. Stackelberg in 1934
- **Two-player sequential-play game:** LEADER and FOLLOWER
- LEADER moves before FOLLOWER - first mover advantage
- **Perfect information:** both agents have perfect knowledge of each others strategy
- **Rationality:** agents act optimally, according to their respective goals



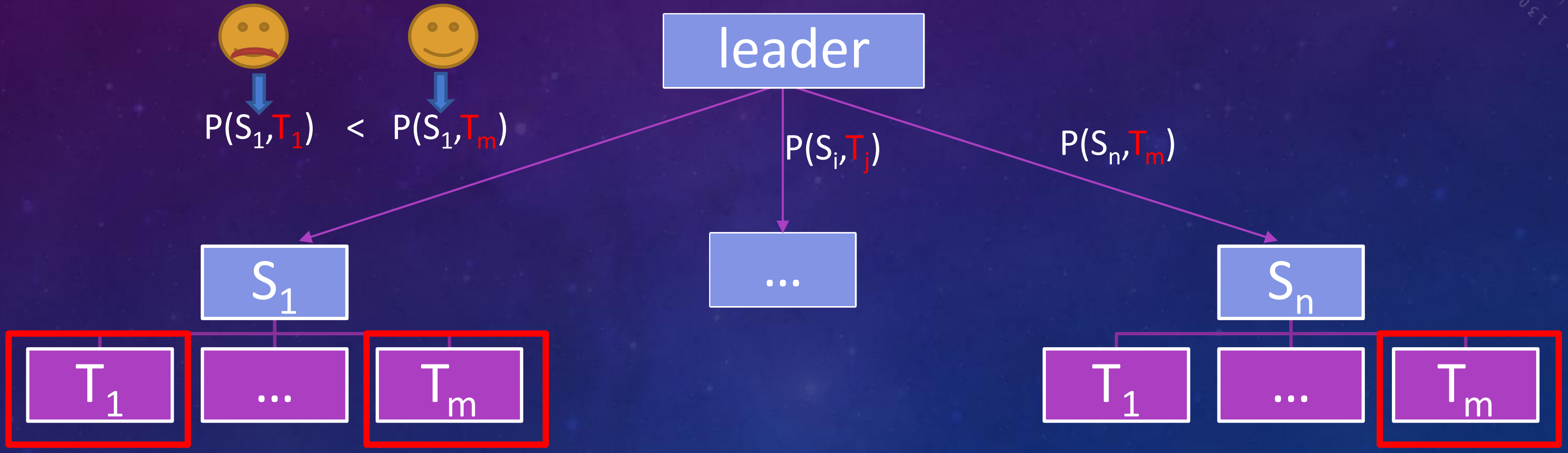
A TWO-PLAYER SETTING



Leader chooses the strategy that maximizes her payoff
Leader anticipates the best response of the follower (backward induction)
Stackelberg equilibrium

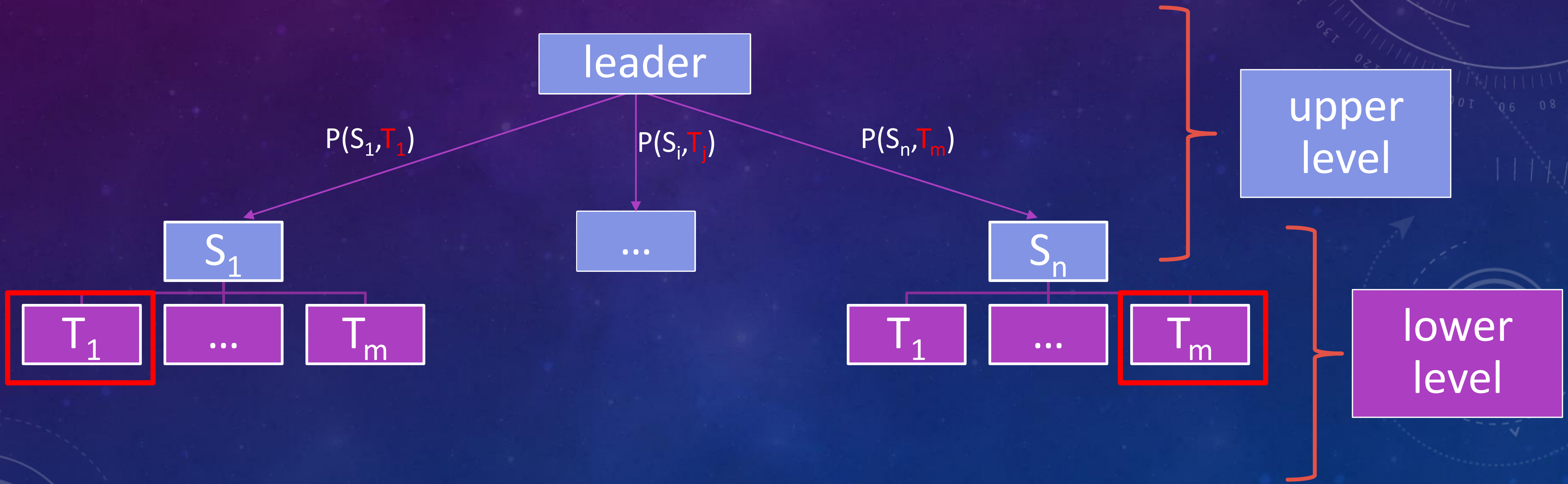
A TWO-PLAYER SETTING: PESSIMISTIC VS OPTIMISTIC?

Pessimistic! imistic!



When multiple strategies of the follower lead to the best response, we can distinguish between “optimistic” and “pessimistic leader”

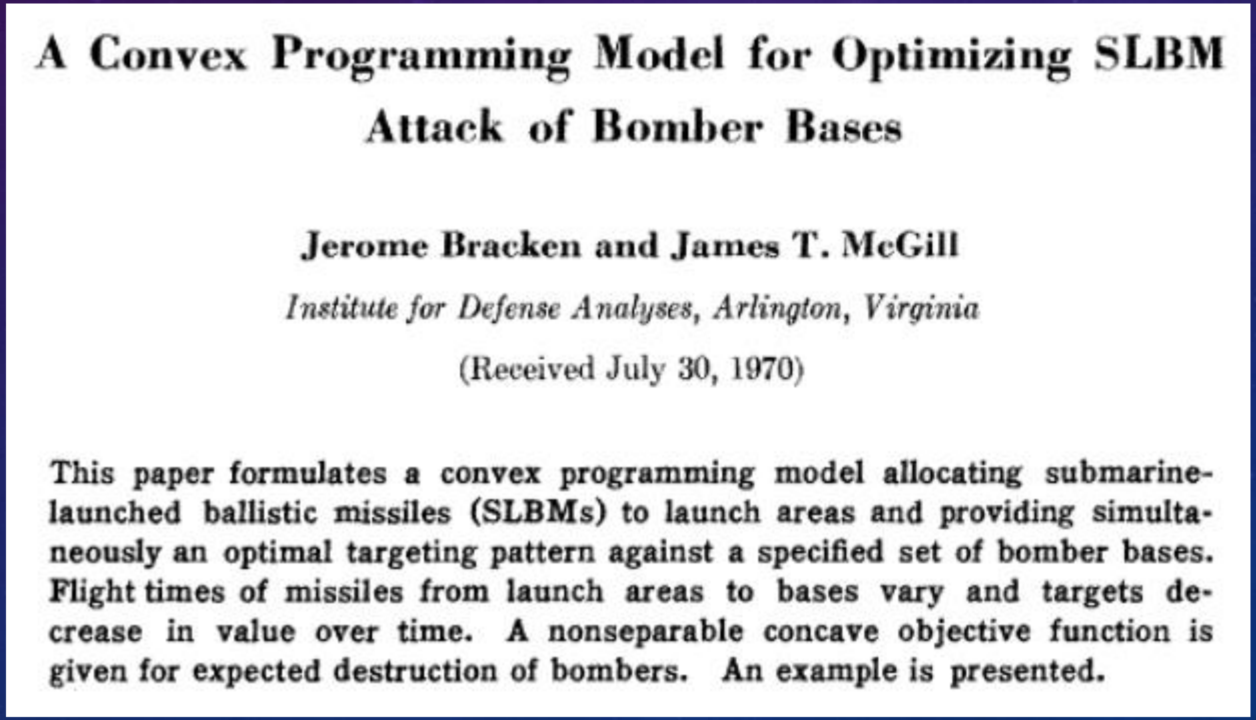
STACKELBERG GAMES



Hierarchical optimization → BILEVEL OPTIMIZATION

STACKELBERG GAMES

- Introduced in economy by v. Stackelberg in 1934
- 40 years later introduced in Mathematical Optimization
  → **Bilevel Optimization**

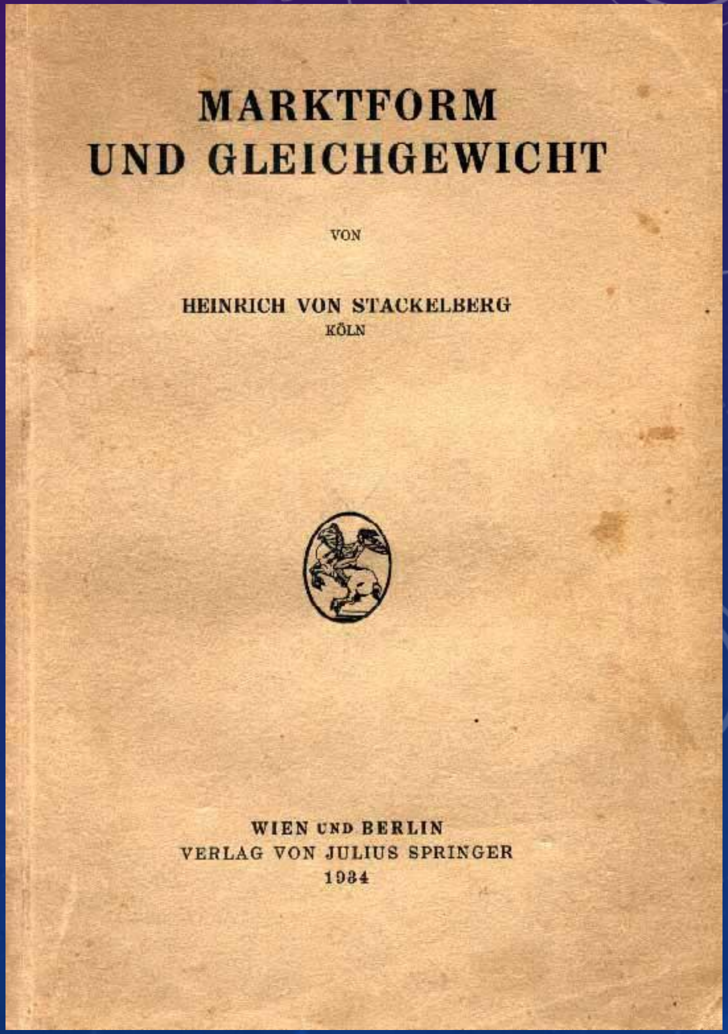

A Convex Programming Model for Optimizing SLBM Attack of Bomber Bases

Jerome Bracken and James T. McGill

Institute for Defense Analyses, Arlington, Virginia

(Received July 30, 1970)

This paper formulates a convex programming model allocating submarine-launched ballistic missiles (SLBMs) to launch areas and providing simultaneously an optimal targeting pattern against a specified set of bomber bases. Flight times of missiles from launch areas to bases vary and targets decrease in value over time. A nonseparable concave objective function is given for expected destruction of bombers. An example is presented.



APPLICATIONS: PRICING

Pricing: operator sets tariffs, and then customers choose the cheapest alternative

- Tariff-setting, toll optimization (Labbé et al., 1998; Brotcorne et al., 2001; Labbé & Violin, 2016)
- Network Design and Pricing (Brotcorne et al., 2008)
- Survey (van Hoesel, 2008)

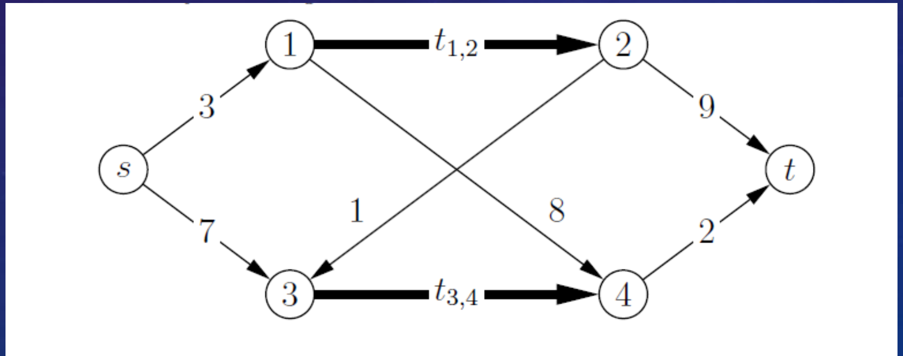


Figure 1: 1-commodity network with two tariff arcs.

APPLICATIONS: INTERDICTION

Canada and the Transcontinental Drug Links

Strategic Forecasting Inc
go to original

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Canadian police conducted several simultaneous raids on suspected drug traffickers in Newfoundland and Quebec provinces Oct. 11, arresting two dozen people and seizing marijuana, cocaine, weapons, cash and property. The drug-trafficking ring, which Canadian authorities believe was operated by the Quebec-based Hell's Angels motorcycle/crime gang, could have smuggled the cocaine into Canada from South America via Mexico and the United States.

MAJOR DRUG SMUGGLING ROUTES THROUGH NORTH AMERICA

The jungles of South America, where cocaine is produced, seem a long way from the St. Lawrence River. Using a sophisticated shipment and distribution network, however, criminal and militant organizations can cover the distance in a few days.

More than 70 members of the Royal Newfoundland Constabulary and Quebec's Provincial Biker Enforcement Unit carried out the raids, which represented the culmination of an 18-monthlong investigation dubbed Operation Roadrunner. The arrests were made near St. John's in Newfoundland and near the towns of Laval and La Tuque in Quebec. In Newfoundland, authorities seized $300,000 in cash, 51 pounds of marijuana and 19 pounds of cocaine, as well as vehicles, weapons and computers. In Quebec, $170,000 and four houses were seized.

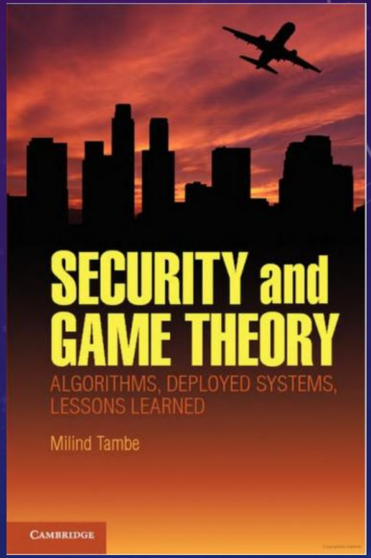
source: banderasnews.com, Oct 2017

APPLICATIONS: INTERDICTION

- Monitoring / halting an adversary's activity
  - Maximum-Flow Interdiction
  - Shortest-Path Interdiction
- Action:
  - Destruction of certain nodes / edges
  - Reduction of capacity / increase of cost
- The problems are NP-hard! Survey (Collado&Papp, 2012)
- Uncertainties:
  - Network characteristics
  - Follower's response

APPLICATIONS: SECURITY GAMES

- Players: DEFENDER (leader) and ATTACKER (follower)
- DEFENDER needs to allocate scare resources to minimize the potential damage caused by ATTACKER
- Leader plays a mixed strategy; Single- or multi-period,multiple followers; imperfect information,…
- Casorrán, Fortz, Labbé, Ordonez, EJOR, 2019.



airport security

poaching

fare evasion

BILEVEL OPTIMIZATION

General bilevel optimization problem

$$
\begin{aligned}
\min_{x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}} \quad & F(x, y) \\
& G(x, y) \leq 0 \\
& y \in \arg \min_{y' \in \mathbb{R}^{n_2}} \{ f(x, y') : g(x, y') \leq 0 \}
\end{aligned}
$$

Leader

Follower

Both levels may involve integer decision variables

Functions can be non-linear, non-convex…

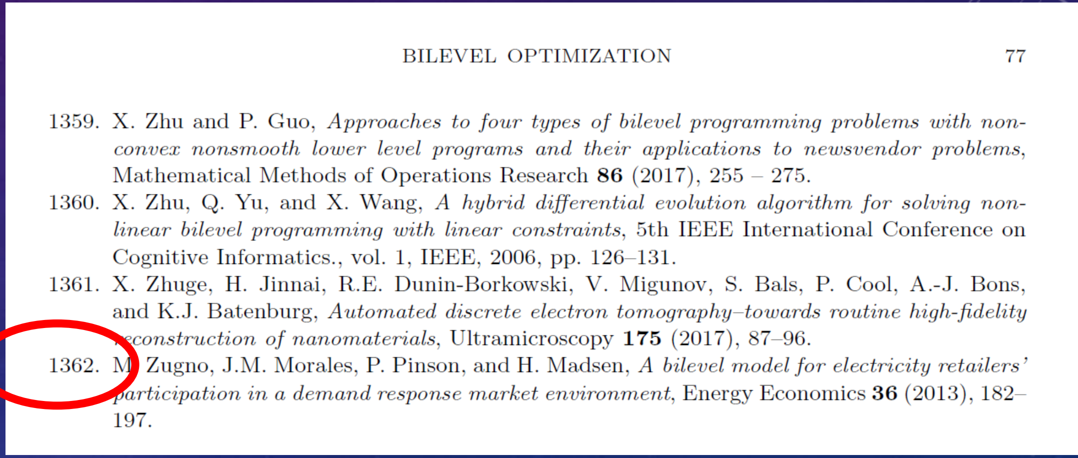
BILEVEL OPTIMIZATION


Stephan Dempe

Bilevel optimization: theory, algorithms and applications

PREPRINT 2018-11

Fakultät für Mathematik und Informatik



1359. X. Zhu and P. Guo, *Approaches to four types of bilevel programming problems with nonconvex nonsmooth lower level programs and their applications to newsvendor problems*, Mathematical Methods of Operations Research **86** (2017), 255 – 275.

1360. X. Zhu, Q. Yu, and X. Wang, *A hybrid differential evolution algorithm for solving nonlinear bilevel programming with linear constraints*, 5th IEEE International Conference on Cognitive Informatics., vol. 1, IEEE, 2006, pp. 126–131.

1361. X. Zhuge, H. Jinnai, R.E. Dunin-Borkowski, V. Migunov, S. Bals, P. Cool, A.-J. Bons, and K.J. Batenburg, *Automated discrete electron tomography–towards routine high-fidelity reconstruction of nanomaterials*, Ultramicroscopy **175** (2017), 87–96.

1362. M. Zugno, J.M. Morales, P. Pinson, and H. Madsen, *A bilevel model for electricity retailers' participation in a demand response market environment*, Energy Economics **36** (2013), 182–197.

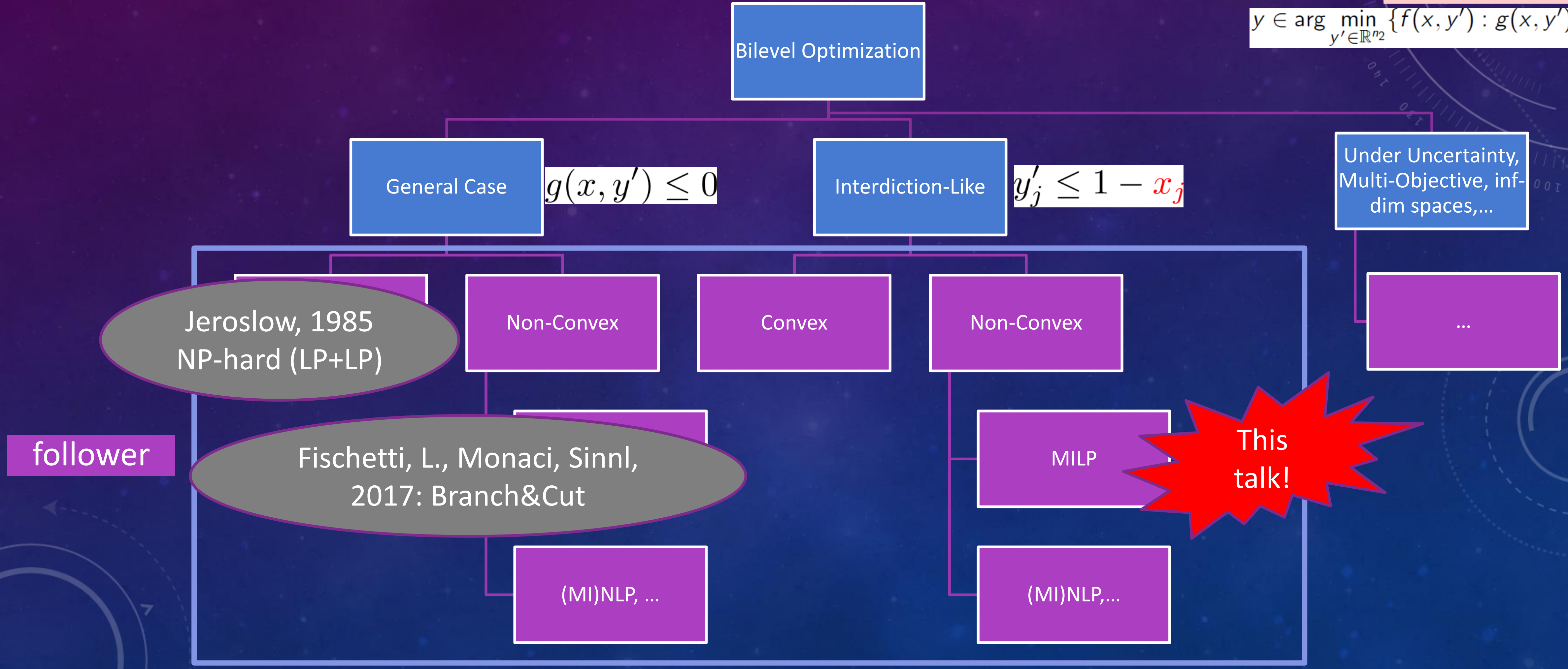
1362 references!

HIERARCHY OF BILEVEL OPTIMIZATION PROBLEMS

$$\min_{x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}} F(x,y)$$

$$G(x,y) \leq 0$$

$$y \in \arg \min_{y' \in \mathbb{R}^{n_2}} \{f(x,y') : g(x,y') \leq 0\}$$

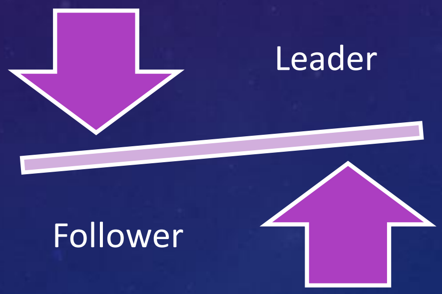


PROBLEMS ADDRESSED TODAY…

FOLLOWER solves a **combinatorial optimization problem** (mostly, an NP-hard problem!).

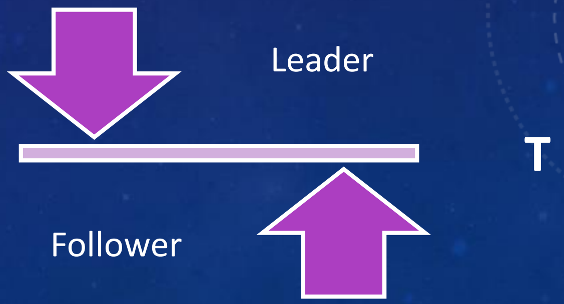
Both agents play pure strategies.

Interdiction Problems: LEADER has a **limited budget** to ”interdict” FOLLOWER by removing some “objects”.



Min-Max Objective

Blocker Problems: LEADER **minimizes the budget** to ”interdict” FOLLOWER by removing some “objects”. The FOLLOWER’s objective should stay below a given threshold **T**



ABOUT OUR JOURNEY

- With **sparse MILP formulations**, we can now solve to optimality:
  - Covering Facility Location (Cordeau, Furini, L., 2018): **20M clients**
    - **Code:** https://github.com/fabiofurini/LocationCovering
  - Competitive Facility Location (L., Moreno, 2017): **80K clients (nonlinear)**
  - Facility Location Problems (Fischetti, L., Sinnl, 2016): 2K x 10K instances
  - Steiner Trees (DIMACS Challenge, 2014): 150k nodes, 600k edges
- **Common to all: Branch-and-Benders-Cut**

Can we exploit sparse formulations along with Branch-and-Cut for bilevel optimization?

BRANCH-AND-INTERDICTION-CUTS FRAMEWORK

- We propose **a generic Branch-and-Interdiction-Cuts framework** to efficiently solve these problems in practice!
- Assuming **monotonicty property** for FOLLOWER: **interdiction cuts** (facet-defining)
- Computationally outperforming state-of-the-art

- Draw a connection to some problems in Graph Theory

BRANCH-AND-INTERDICTION-CUT

A GENTLE INTRODUCTION

BILEVEL KNAPSACK WITH INTERDICTION CONSTRAINTS

$$\min_{x \in \mathbb{B}^n} \; p^T y$$

$$v^T x \leq C_l$$

where y solves the follower problem

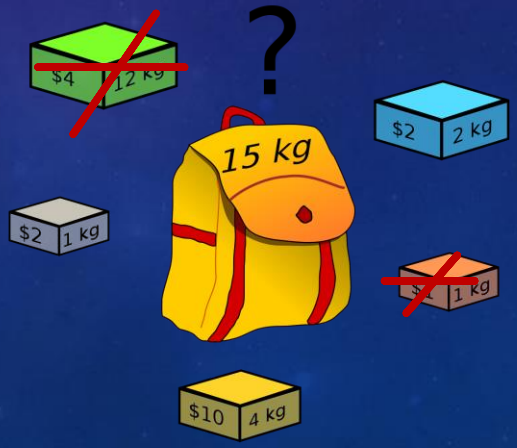
$$\max_{y \in \mathbb{B}^n} \; p^T y$$

$$w^T y \leq C_f$$

$$y_i \leq 1 - x_i \quad i = 1, \ldots, n$$

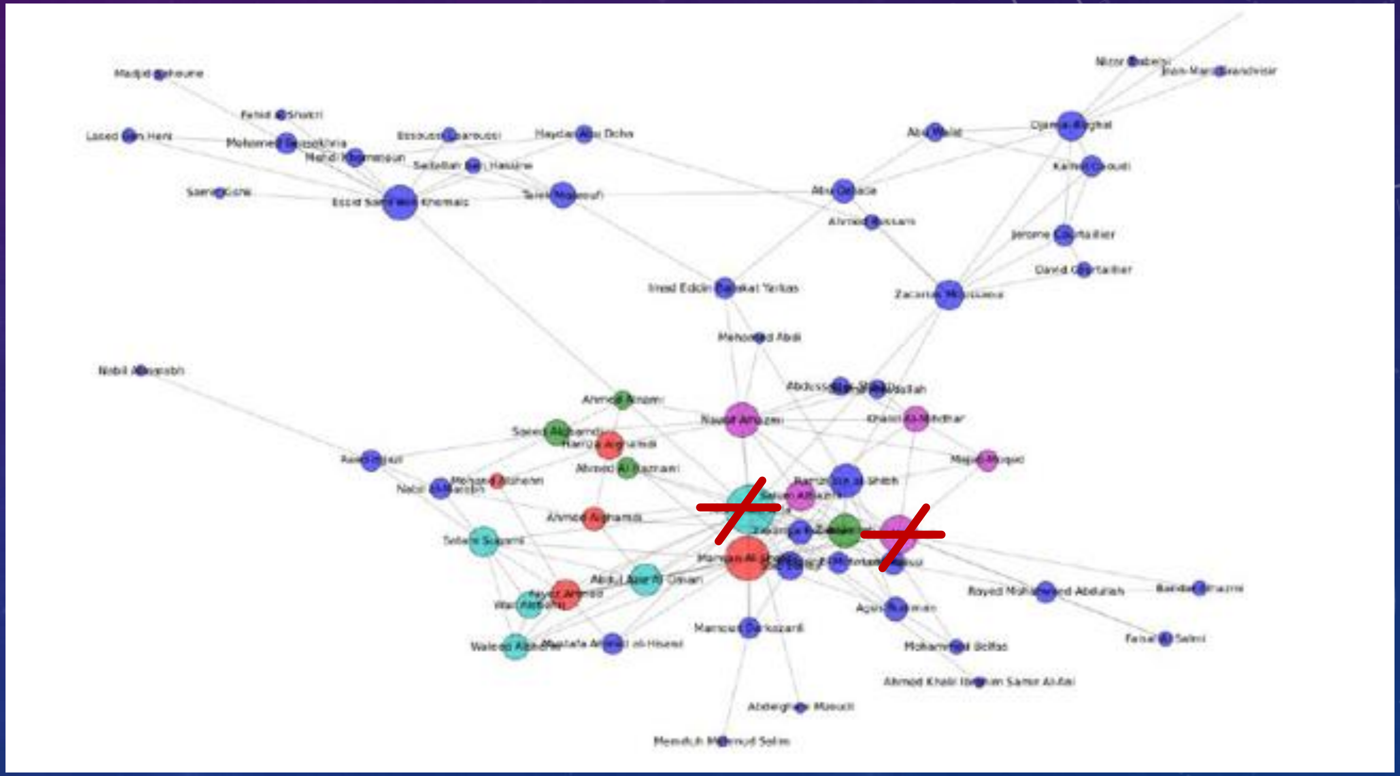
Marketing Strategy Problem (De Negre, 2011)

Companies A (leader) and B (follower).
Items are geographic regions.
Cost and benefit for each target region.
A dominates the market: whenever A and B target the same region, campaign of B is not effective



THE CLIQUE INTERDICTION PROBLEM

- Marc Sageman (“Understanding terror networks”) studied the “Hamburg cell” network (172 terrorists): social ties very strong in densely connected networks
- Cliques
- Given an interdiction budget k, which k nodes to remove from the network so that the remaining maximum clique is smallest possible?



THE CLIQUE INTERDICTION PROBLEM

$$\min_{x \in \mathbb{B}^n} \; p^T y$$

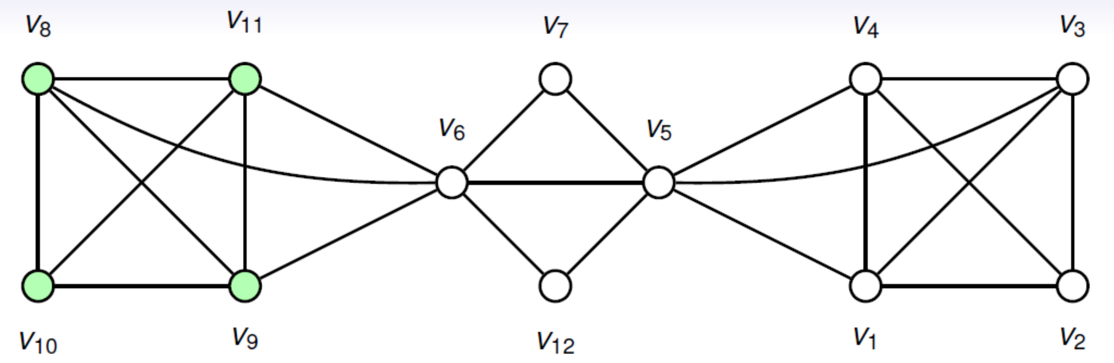
$$v^T x \leq C_l$$

where y solves the follower problem

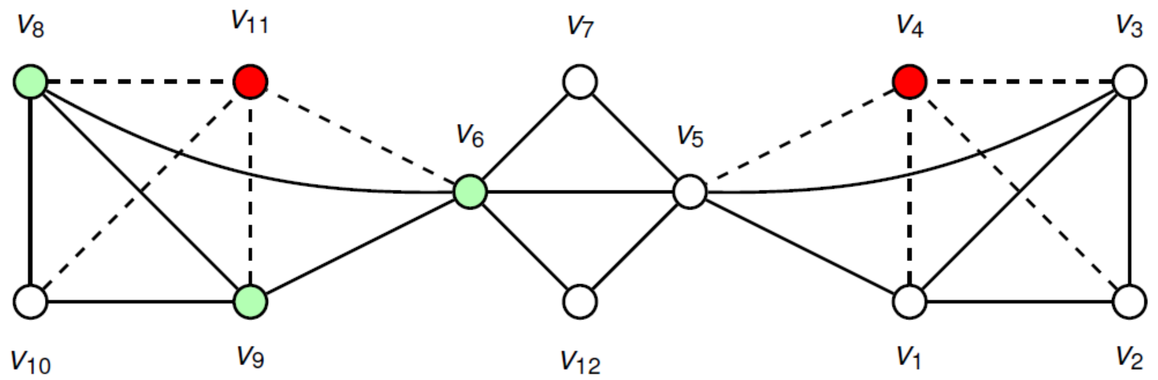
$$\max_{y \in \mathbb{B}^n} \; p^T y$$

y is a clique $\quad (y \in Y)$

$$y_i \leq 1 - x_i \quad i = 1, \ldots, n$$



The clique number is $\omega(G) = 4$ ($K_2 = \{v_8, v_9, v_{10}, v_{11}\}$)



An optimal interdiction strategy with $k = 2$ ($\omega(G[V \setminus \{v_4, v_{11}\}]) = 2$)

GENERAL SETTING

$$
\begin{aligned}
&\min d^T y \\
&\quad v^T x \le C_l \\
&\quad x \in X \\
&y \in \arg\max\{d^T y : \\
&\qquad y_i \le 1 - \color{red}{x_i}, \quad \color{red}{i \in N} \\
&\qquad y \in Y\} \\
&\quad x_i \text{ binary}, \quad i \in N
\end{aligned}
$$

$$
y_i = \begin{cases} 1 & \text{if } i \text{ belongs to the follower's solution} \\ 0 & \text{otherwise} \end{cases} \qquad i \in N.
$$

$$
x_i = \begin{cases} 1 & \text{if } i \text{ is interdicted} \\ 0 & \text{otherwise} \end{cases} \qquad i \in N.
$$

$$
\begin{aligned}
&\min w \\
&\quad v^T x \le C_l \\
&\quad x \in X \\
&w \ge \max\{d^T y : \\
&\qquad y_i \le 1 - \color{red}{x_i}, \quad \color{red}{i \in N} \\
&\qquad y \in Y\} \\
&\quad x_i \text{ binary}, \quad i \in N
\end{aligned}
$$

Value Function

$= \Phi(\mathbf{x})$

VALUE FUNCTION REFORMULATION

$$\min_{x \in X, w \in \mathbb{R}} \quad w$$

$$w \geq \Phi(x)$$

$$v^T x \leq C_l$$

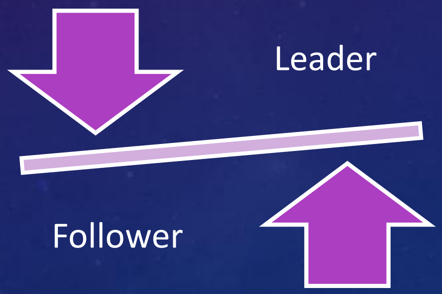
$$x_i \text{ binary}, \quad i \in N$$

$$\min_{x \in X} b^T x$$

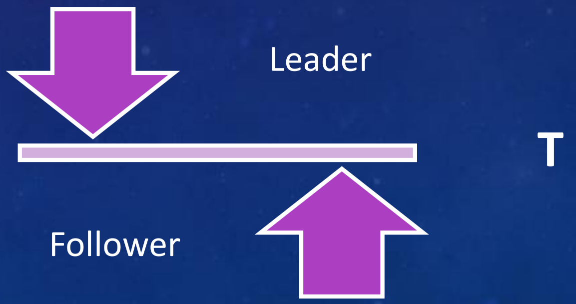
$$T \geq \Phi(x)$$

$$x_i \text{ binary}, \quad i \in N$$

INTERDICTION: Min-max



BLOCKING: Min-num or Min-sum



VALUE FUNCTION REFORMULATION

$$\min_{x \in X, w \in \mathbb{R}} \quad w$$
$$w \geq \Phi(x)$$
$$v^T x \leq C_l$$
$$x_i \text{ binary}, \quad i \in N$$

$$\min_{x \in X} b^T x$$
$$T \geq \Phi(x)$$
$$x_i \text{ binary}, \quad i \in N$$

GENERAL IDEA:

- Benders-Like Reformulation: y variables are projected out!
- If function $\Phi(x)$ could be "convexified" (using linear functions in x), we would obtain an MILP!
- To be solved in a branch-and-cut fashion

HOW TO CONVEXIFY THE VALUE FUNCTION?

CONVEXIFICATION

Observation: Given x, for the optimal follower's response it holds:

$$x_j + y_j \leq 1 \quad \Rightarrow \quad x_j y_j = 0 \qquad j \in N$$

Instead of solving:

$$\Phi(x) = \max_{y \in \mathbb{R}^{n_2}} d^T y$$
$$0 \leq y_j \leq 1 - x_j, \quad \forall j \in N$$
$$y \in Y$$

$$Y = \{y \in \mathbb{R}^{n_2} : \quad Q\, y \leq q_0,$$
$$y_j \text{ integer } \forall j \in J_y\}.$$

Wood (2011) proposes to move x into the objective function and find the penalties M_j, such that we can equivalently solve:

$$\Phi(x) = \max_{y \in \mathbb{R}^{n_2}} \{d^T y - \sum_{j \in N} M_j x_j y_j \quad y \in Y\} = \max_{\hat{y} \in \mathrm{conv}(Y)} \{d^T \hat{y} - \sum_{j \in N} M_j x_j \hat{y}_j\}$$

CONVEXIFICATION → BENDERS-LIKE REFORMULATION

Benders–Like Reformulation

$$\min_{x \in \mathbb{R}^{n_1}, w \in \mathbb{R}} \quad w$$

$$w \geq d^T \hat{y} - \sum_{j \in N} M_j x_j \hat{y}_j \qquad \forall \hat{y} \in \hat{Y}$$

$$Ax \leq b$$

$$x_j \text{ integer}, \qquad \forall j \in J_x$$

$$x_j \text{ binary}, \qquad \forall j \in N.$$

The choice of M_j is crucial:

- If FOLLOWER solves an LP: Wood (2011), M_j is the upper bound of the dual variable.
- If FOLLOWER solves the KNAPSACK PROBLEM: Caprara et al. (2016), De Negre (2011), $M_j = d_j$.
- In general: **OPEN QUESTION**.

IF THE FOLLOWER SATISFIES MONOTONICITY PROPERTY…

Downward Monotonicity: $Q \geq 0$
If $\hat{y}$ is a feasible follower solution and y' satisfies integrality constraints and $0 \leq y' \leq \hat{y}$, then y' is *also feasible*.

$$Y = \{y \in \mathbb{R}^{n_2} : Q\, y \leq q_0, \quad y_j \text{ integer } \forall j \in J_y\}.$$

Theorem:
For Interdiction Games with Monotonicity $M_j = d_j$, i.e., we have:

$$\min_{x \in \mathbb{R}^{n_1}, w \in \mathbb{R}} w$$
$$w \geq \sum_{j \in N} d_j \hat{y}_j (1 - x_j) \qquad \forall \hat{y} \in \hat{Y}$$
$$Ax \leq b$$
$$x_j \text{ integer}, \qquad \forall j \in J_x$$
$$x_j \text{ binary}, \qquad \forall j \in N.$$

- **max-knapsack (set packing)**
- **max-clique**
- **max-relaxed-clique (s-plex: degree, s-clique: distance, s-bundle: connectivity)**

Fischetti, Ljubic, Monaci, Sinnl , IJOC 2019

SOME THEORETICAL PROPERTIES...

Under **Downward Monotonicity,** $Q \geq 0$:

- interdiction cuts are facet-defining under mild conditions [1],
- interdiction cuts can be efficiently down-lifted [2],
- specific pre-processing and dominance rules can be developed [1,2]

References:

[1] Furini, L., Martin, San Segundo: The Maximum Clique Interdiction Problem, European Journal of Operational Research 277(1):112-127, 2019

[2] Fischetti, L., Monaci, Sinnl: Interdiction Games and Monotonicity, with Application to Knapsack Problems, INFORMS Journal on Computing, 31(2):390-410, 2019

SLIDE "NOT TO BE SHOWN"

The follower:

Interdiction Cuts WORK WELL EVEN IF FOLLOWER HAS MORE DECISION VARIABLES, AS LONG AS MONOTONOCITY HOLDS FOR INTERDICTED VARIABLES

Downward Monotonicity: Assu… ≥ 0

"if $\hat{y} = (\hat{y}_N, \hat{y}_R)$ is a feasible follower for a given x and $y' = (y'_N, \hat{y}_R)$ satisfies integrality constraints and $0 \leq y'_N \leq \hat{y}_N$, then y' is **also feasible** for x".

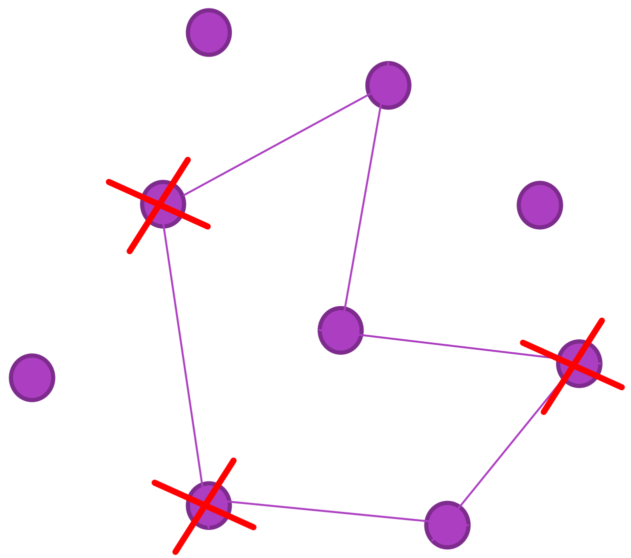
THE RESULT CAN BE FURTHER GENERALIZED

Relevant Operations Research applications. Two companies competing at the market for customers.

- LEADER: established on the market,
- FOLLOWER: a newcomer who wants to disrupt the market.

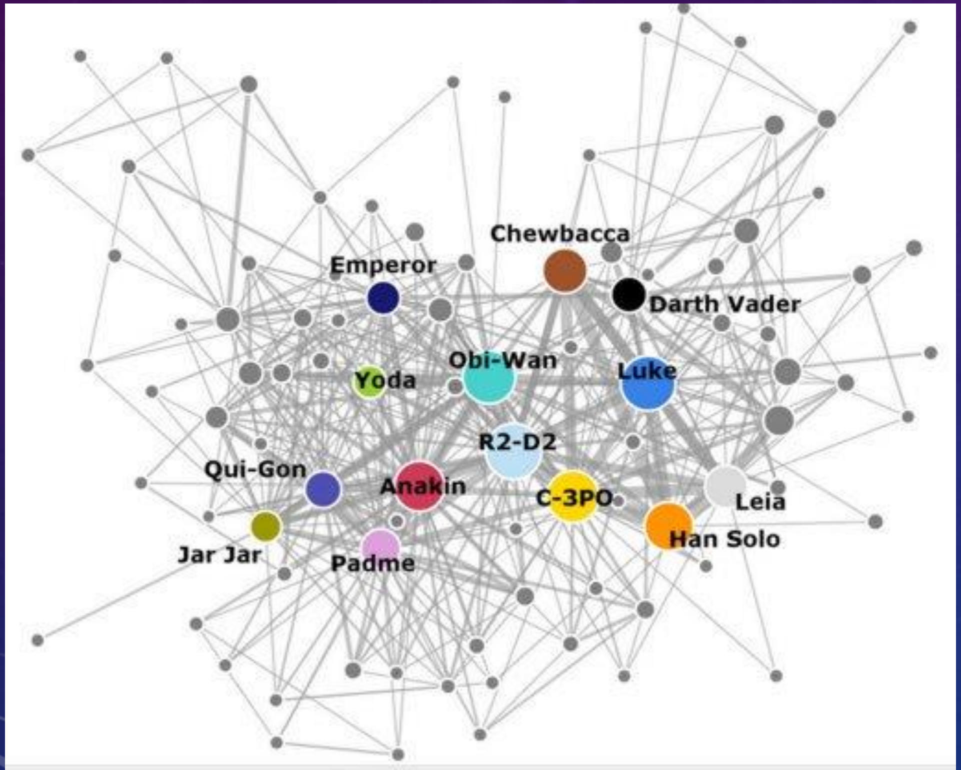
LEADER wants to keep the customers by providing them coupons, vouchers. FOLLOWER is solving a profit-maximization problem:

- **NETWORK DESIGN**: prize-collecting Steiner tree
- **LOGISTICS**: orienteering problems
- **FACILITY LOCATION**: profit maximization variant

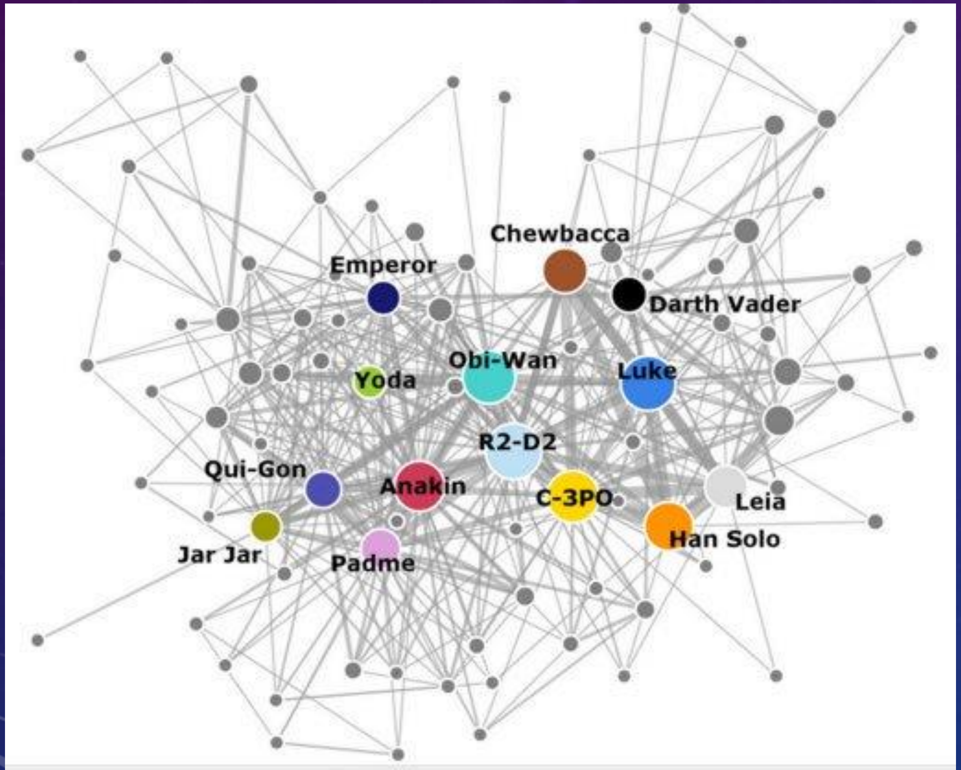


Fischetti, Ljubic, Monaci, Sinnl, IJOC (2019)

CRITICAL NODE/EDGE DETECTION PROBLEMS



CRITICAL NODE/EDGE DETECTION PROBLEMS



Node Centrality

- Degree-, betweenness-, distance-, eigenvalue-centrality,...

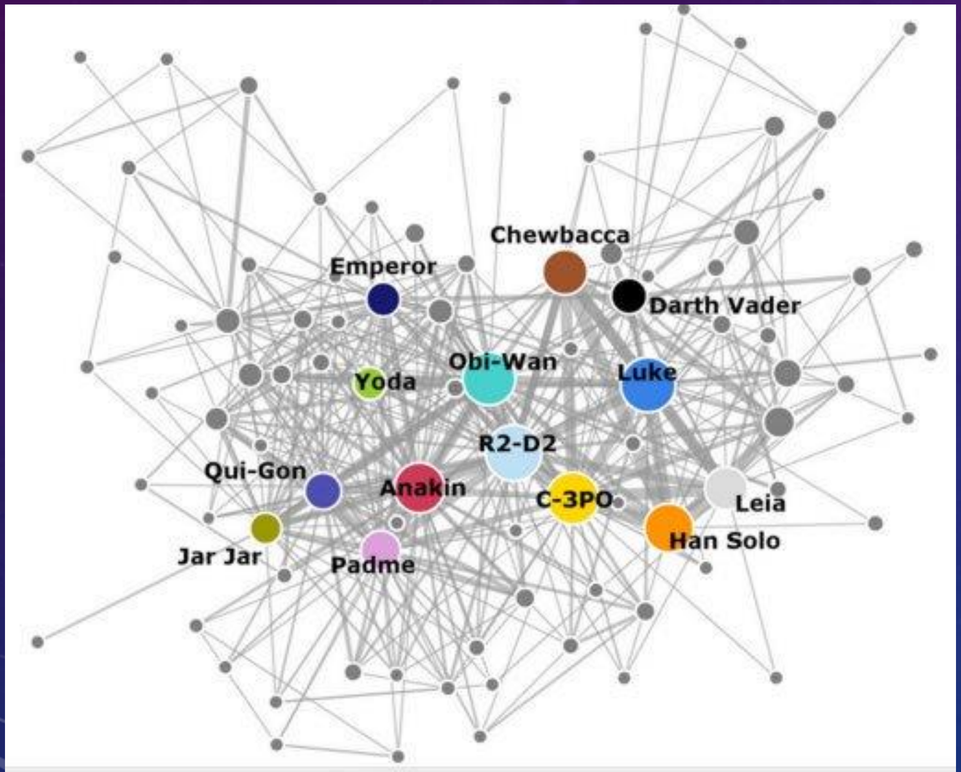
Individual vs **Collective** Centrality?

Greedy selection of the most central k nodes is **suboptimal!** (Shen, Smith, Goli, 2012)

Evaluation of $\binom{n}{k}$ all possible combinations is intractable!

$$\binom{500}{10} = 2.4581058880189 \times 10^{20}.$$

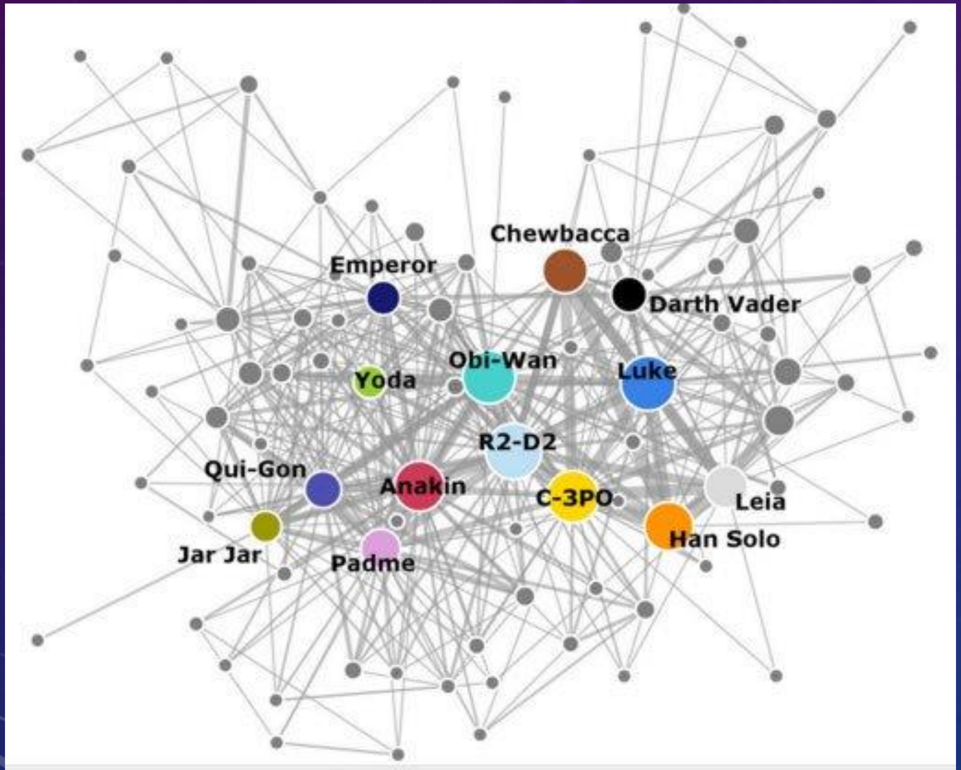
CRITICAL NODE/EDGE DETECTION PROBLEMS



Connectivity-based Centrality Measures:

- the number of connected components
- the size of a largest connected component
- the number of pairwise disconnected node pairs
- the number of edges needed to reconnect the graph
- the size of the largest (relaxed) clique, ...

CRITICAL NODE/EDGE DETECTION PROBLEMS

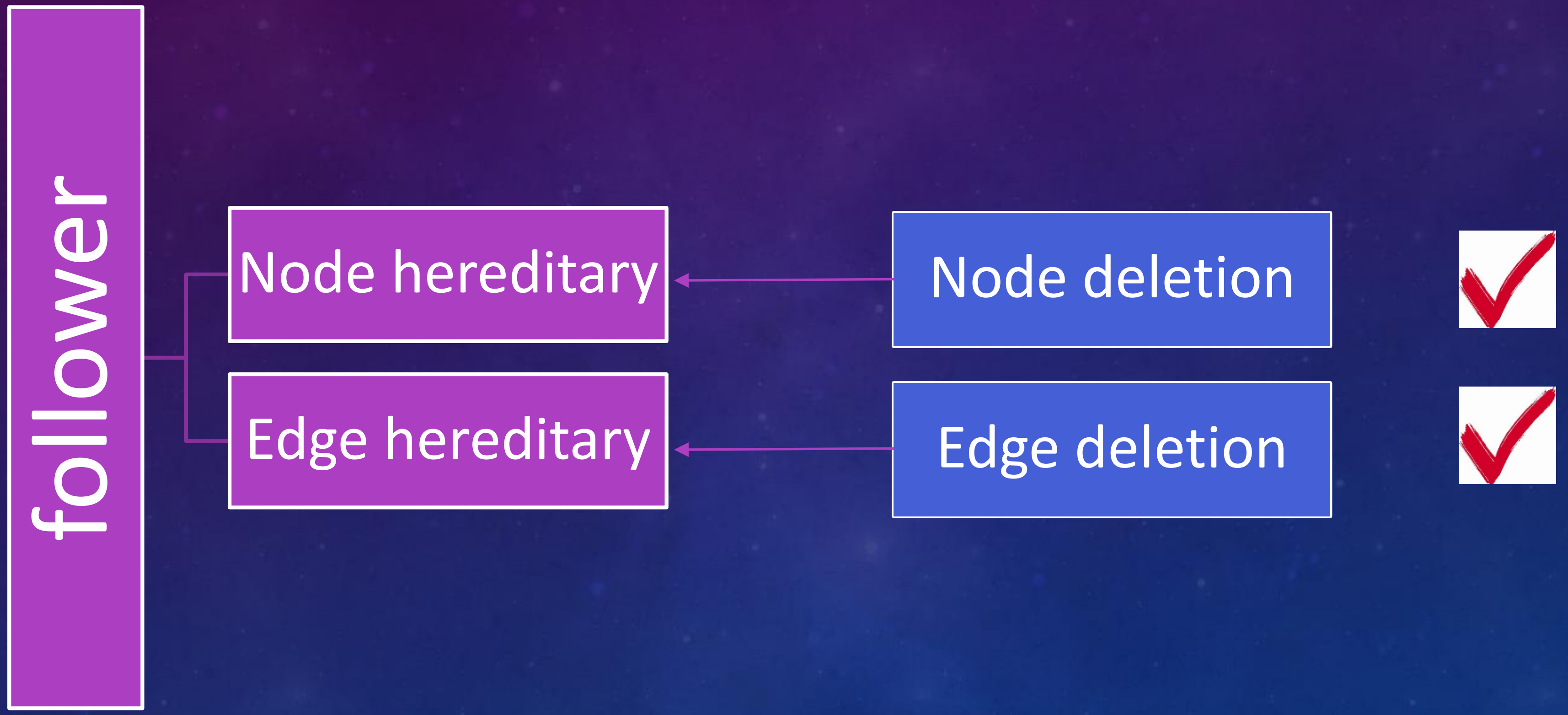


Stackelberg Games:

- LEADER removes edges/nodes
- FOLLOWER optimizes the connectivity measure

Sparse MIP models
vs
Very large extended formulations

HEREDITARY PROPERTY OF THE FOLLOWER

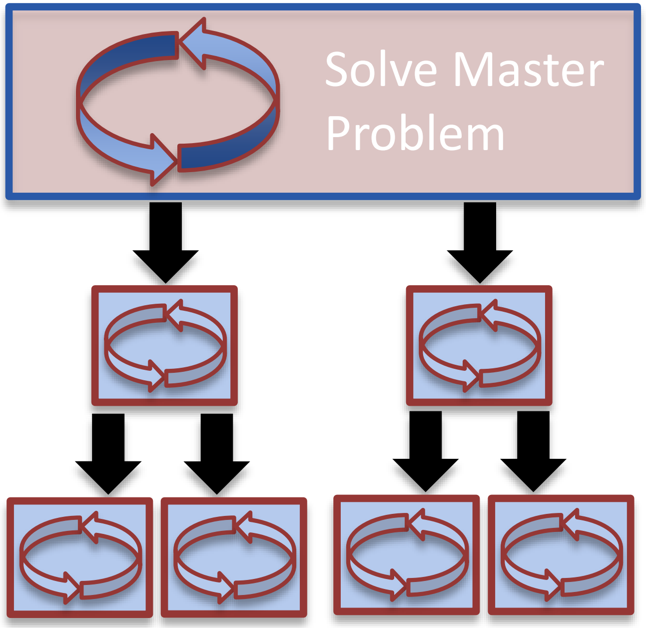


Otherwise: a slightly extended formulation is needed (cf. k-vertex cut)

BRANCH-AND-INTERDICTION-CUT IMPLEMENTATION

A CAREFUL BRANCH-AND-INTERDICTION-CUT DESIGN

- **Separation:** finding the best FOLLOWER's response for a given x^*. **NP–hard, in general**.
- A good **balance** between "lazy cut separation" (integer points only) and "user cut separation" (fractional points).
- **Crucial:** **specialized procedures/algorithms** for FOLLOWER's sub-problem (if available).
- **Combinatorial** algorithms for **LOWER** and **UPPER BOUNDS**.
- Efficient **PREPROCESSING** techniques.
- Under **monotonicity property**: Interdiction cuts are **facet–defining** or could be lifted, otherwise.
- Resulting in general in **strong LP–relaxation bounds**.



Branch-and-Interdiction-Cut

MAX-CLIQUE-INTERDICTION: LARGE-SCALE NETWORKS

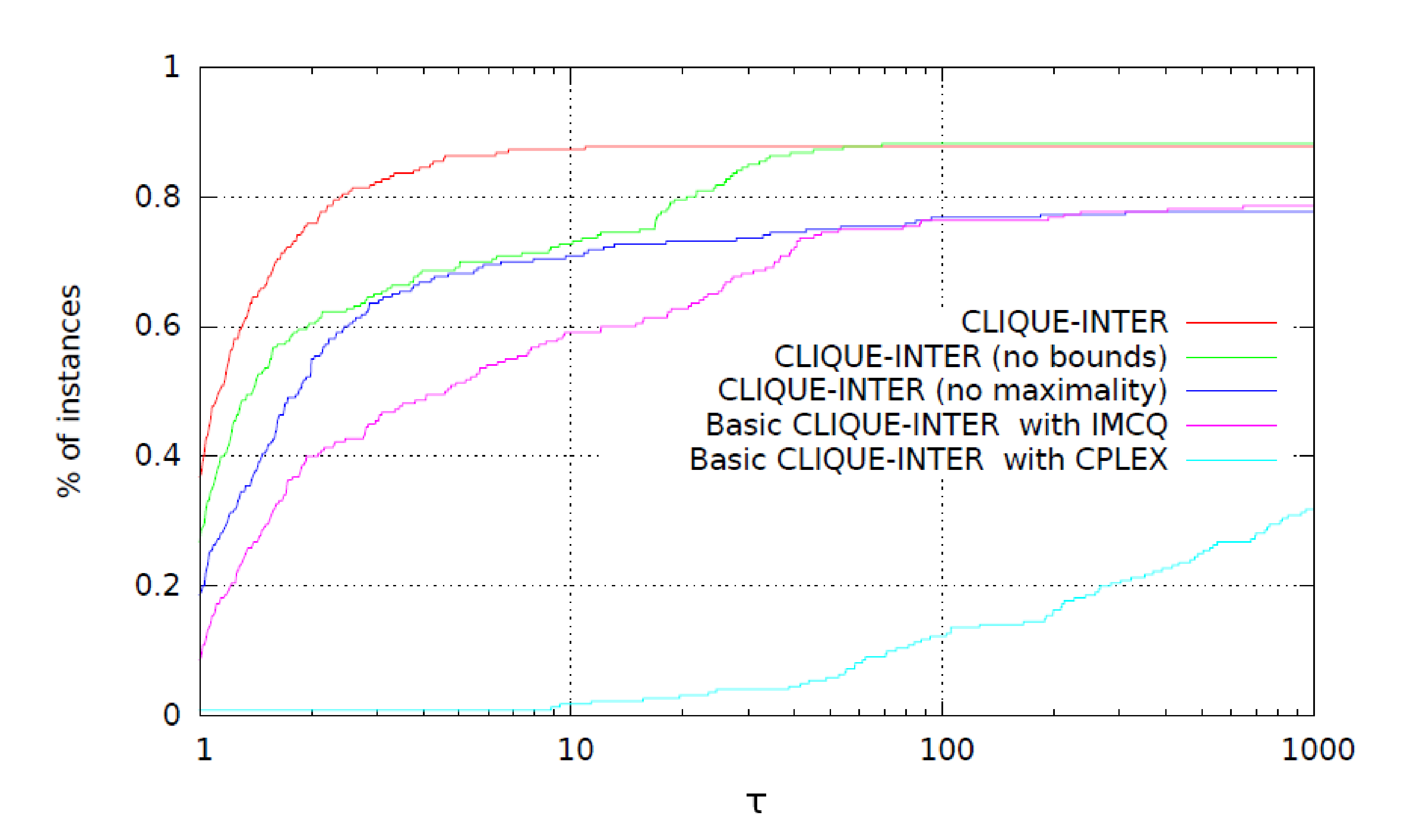
	Max-Clique Solver San Segundo et al. (2016)			$k = \lceil 0.005 \cdot \lvert V \rvert \rceil$		$k = \lceil 0.01 \cdot \lvert V \rvert \rceil$	
	$\lvert V \rvert$	$\lvert E \rvert$	ω [s]	t [s]	$\lvert V_p \rvert$	t [s]	$\lvert V_p \rvert$
socfb-UIllinois	30,795	1,264,421	0.5	24.4	10,456	41.6	8290
ia-email-EU	32,430	54,397	0.0	0.6	30,375	0.5	29,212
ia-enron-large	33,696	180,811	0.0	2.2	27,791	29.5	26,651
socfb-UF	35,111	1,465,654	0.3	17.8	14,264	87.8	10,708
socfb-Texas84	36,364	1,590,651	0.3	24.6	10,706	74.3	8,704
fe-tooth	78,136	452,591	0.5	18.9	7	19.0	7
sc-pkustk11	87,804	2,565,054	1.1	70.7	2,712	57.1	2,712
ia-wiki-Talk	92,117	360,767	0.2	49.2	72,678	87.4	72,678
sc-pkustk13	94,893	3,260,967	1.3	724.9	2,360	879.2	2,354

eliminated by preprocessing

#variables

Furini, Ljubic, Martin, San Segundo, EJOR,2019

B&IC INGREDIENTS



COMPARISON WITH THE STATE-OF-THE-ART MILP BILEVEL SOLVER

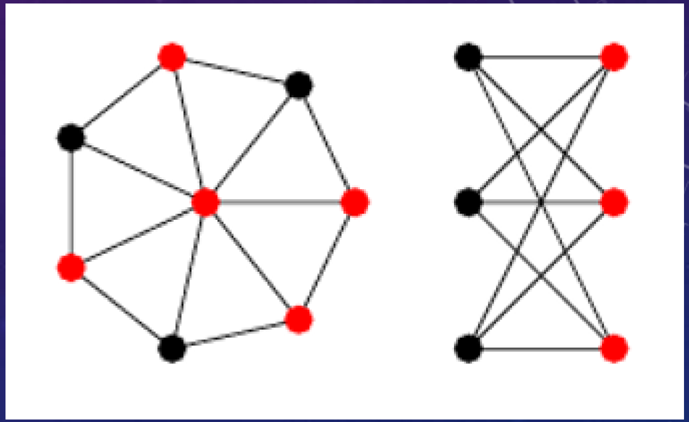
		Branch-and-Interdiction-Cut				Generic B&C for Bilevel MILPs (Fischetti, Ljubic, Monaci, Sinnl, 2017)			
$\lvert V \rvert$	#	# solved	time	exit gap	root gap	# solved	time	exit gap	root gap
50	44	**44**	**0.01**	-	0.16	28	68.58	6.44	8.50
75	44	**44**	**1.45**	-	0.41	14	120.19	9.47	10.91
100	44	**37**	**9.30**	1.00	0.98	7	164.42	12.65	13.11
125	44	**35**	**13.43**	1.33	1.20	2	135.33	13.88	14.73
150	44	**33**	**27.23**	1.91	1.43	1	397.52	16.42	16.39

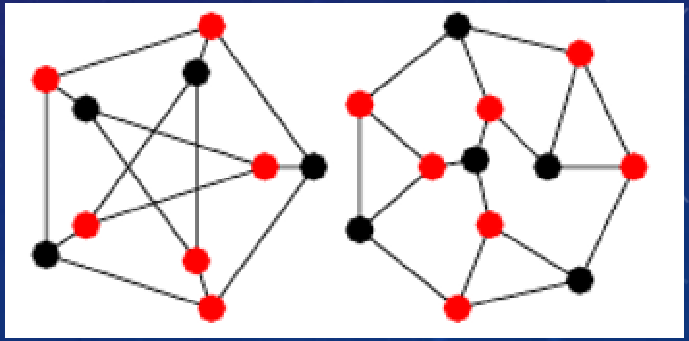
AND WHAT ABOUT GRAPH THEORY?

A WEIRD EXAMPLE

- **Property:** A set of vertices is a **vertex cover** if and only if its complement is an independent set

- **Vertex Cover as a Blocking Problem:**
  - LEADER: interdicts (removes) the nodes.
  - FOLLOWER: maximizes the size of the largest connected component in the remaining graph.
  - Find the smallest set of nodes to interdict, so that FOLLOWER's optimal response is at most one.



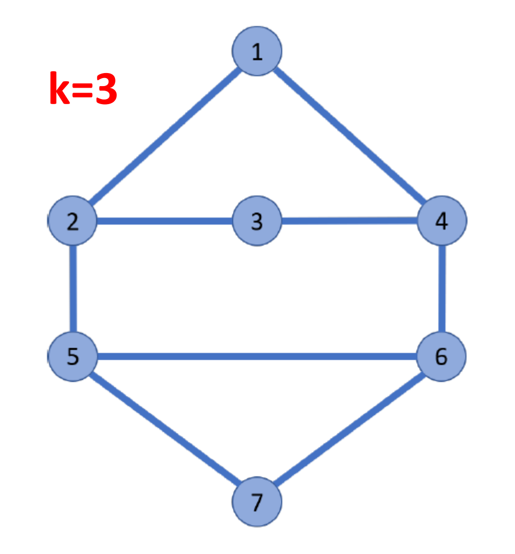


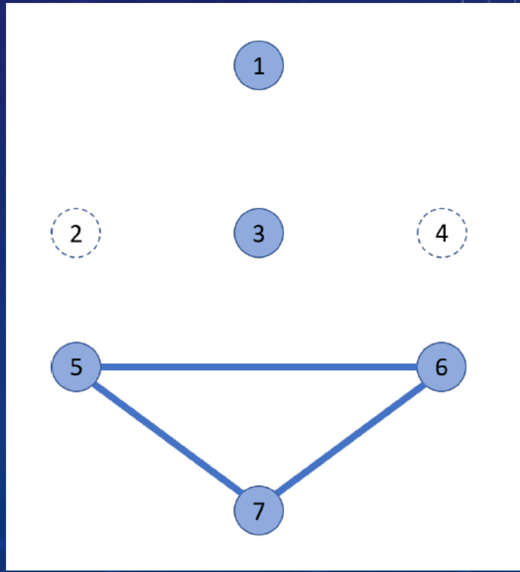
THE K-VERTEX-CUT PROBLEM

- A set of vertices is a **vertex k–cut** if upon its removal the graph contains at least k connected components.
- **The k Vertex–Cut Problem:**
  Find a vertex k-cut of minimum cardinality/weight.

Open question:
ILP formulation in the natural space of variables

- Influential nodes in a diffusion model for social networks, Kempe et al. (2005)
- Decomposition method for linear equation systems, e.g. GCG solver (Bastubbe, Lübbecke, 2017)

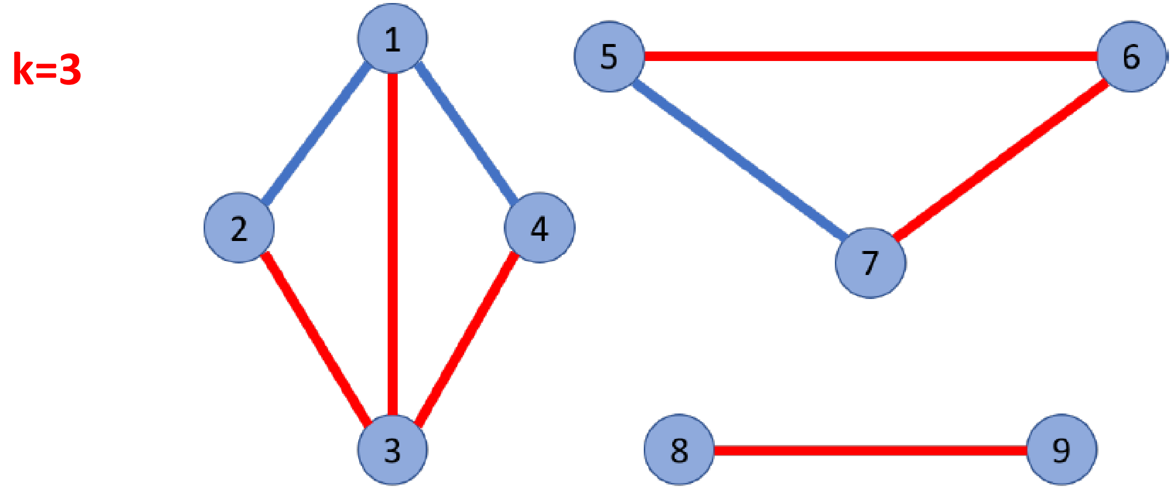




K-VERTEX-CUT

Property: A graph G has at least k (not empty) components if and only if any cycle-free subgraph of G contains at most $|V| - k$ edges.

Example with $|V| = 9$ and $k = 3$:



K-VERTEX-CUT

Stackelberg game:

- LEADER: searches the min-weight subset of nodes V_0 to delete;
- FOLLOWER maximizes the size of the cycle-free subgraph on the interdicted graph.
- Solution of the LEADER is feasible iff optimal FOLLOWER's response is at most $T = |V| - |V_0| - k$.

$$\min \sum_{v \in V} w_v x_v$$

$$\Phi(x) \leq |V| - \sum_{v \in V} x_v - k$$

$$x_v \in \{0,1\} \qquad v \in V.$$

K-VERTEX-CUT: BENDERS-LIKE REFORMULATION

The following **Natural Space Formulation**, is a valid model for the k-vertex cut problem:

$$\min \sum_{v \in V} x_v$$

$$\sum_{v \in V} [\deg_T(v) - 1] x_v \geq k - |V| + |E(T)| \qquad T \in \mathcal{T},$$

$$x_v \in \{0, 1\} \qquad v \in V.$$

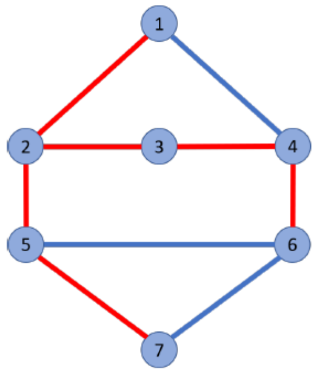
where $\mathcal{T}$ is the set of all (maximal) cycle-free subgraphs of G.

Separation of "interdiction cuts" is polynomial.

Furini, Ljubic, Malaguti, Paronuzzi (2018)

K-VERTEX-CUT: BENDERS-LIKE REFORMULATION

$$\sum_{v \in V} [\deg_T(v) - 1] x_v \geq k - |V| + |E(T)|$$

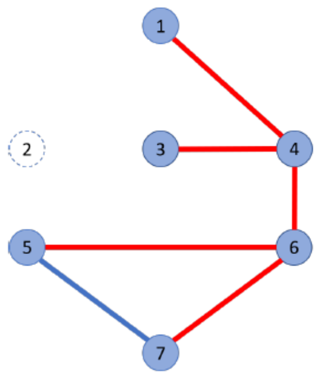


$$2x_2 + x_4 + x_5 \geq 2$$

Furini, Ljubic, Malaguti, Paronuzzi (2018)

K-VERTEX-CUT: BENDERS-LIKE REFORMULATION

$$\sum_{v \in V} [\deg_T(v) - 1] x_v \geq k - |V| + |E(T)|$$

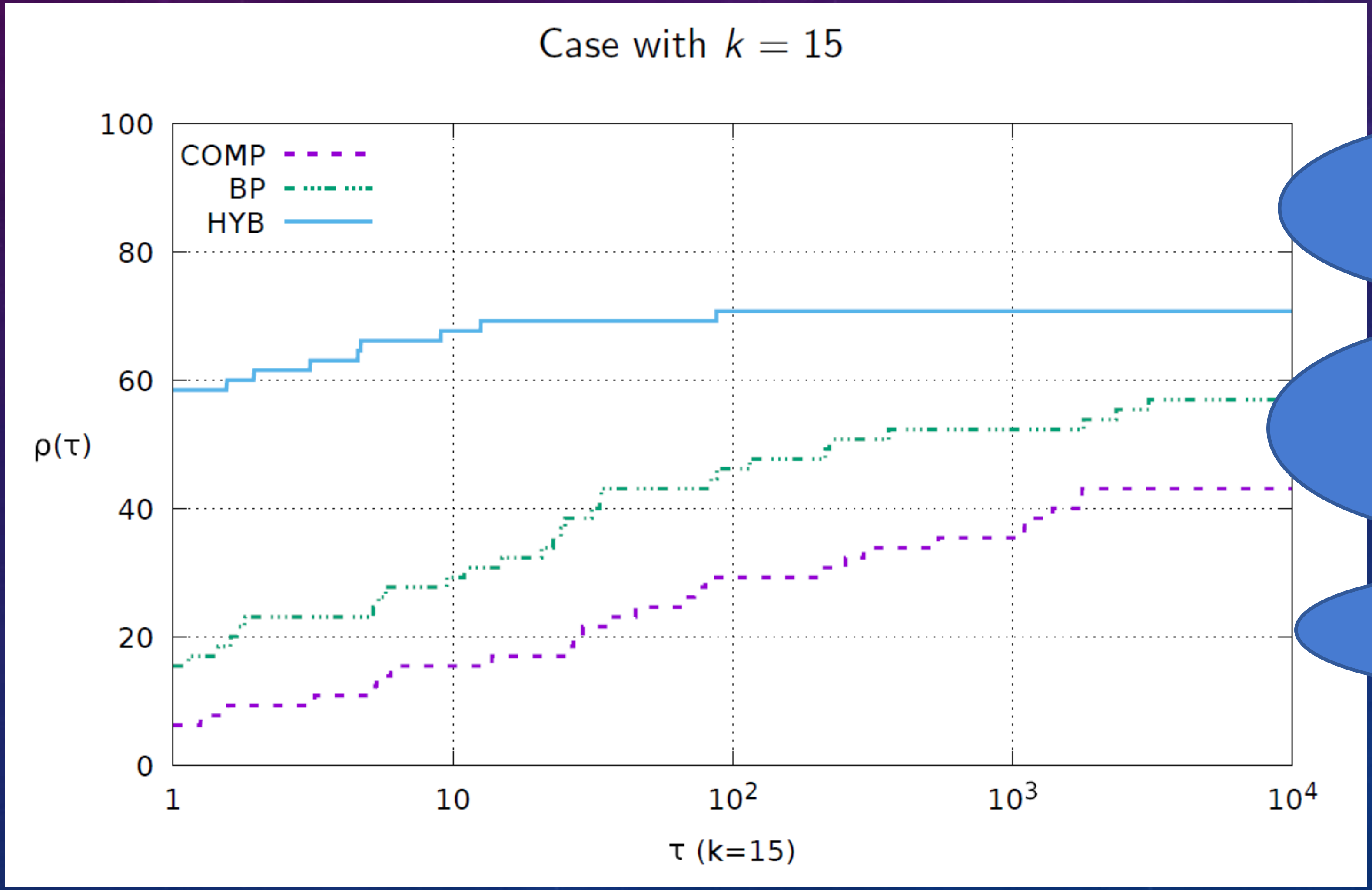


$$2x_2 + x_4 + x_5 \geq 2$$

$$-x_2 + 2x_4 + 2x_6 \geq 1$$

Furini, Ljubic, Malaguti, Paronuzzi (2018)

COMPUTATIONAL PERFORMANCE



Branch-and-Interdiction-Cut

Furini et al. (2018) Prev. STATE-OF-THE-ART

Compact model

CONCLUSIONS.

AND SOME DIRECTIONS FOR THE FUTURE RESEARCH.

TAKEAWAYS

- Bilevel optimization: very difficult!

- **Branch-and-Interdiction-Cuts** can work very well in practice:
  - Problem reformulation in the **natural space of variables** („**thinning out**“ the heavy MILP models)
  - **Tight „interdiction cuts“** (**monotonicity** property)
  - **Crucial:** Problem-dependent (combinatorial) separation strategies, preprocessing, combinatorial poly-time bounds

- Many **graph theory problems** (node-deletion, edge-deletion) could be solved efficiently, when **approached from the bilevel-perspective**

DEALING WITH BILEVEL MILPS

- Check first: is it an interdiction/blocker problem?
- Does it satisfy monotonicity property?
- Graph problems: Is the follower's subproblem hereditary (wrt nodes/edges)?

- If yes, go for a branch-and-interdiction cut.
- Otherwise, try our GENERAL PURPOSE BILEVEL MILP SOLVER:

https://msinnl.github.io/pages/bilevel.html

CHALLENGING DIRECTIONS FOR FUTURE RESEARCH

- **Bilevel Optimization:** a better way of **integrating customer behaviour** into decision making models

- Generalizations of **Branch-and-Interdiction-Cuts** for:
  - **Non-linear** follower functions
  - **Submodular** follower functions
  - Interdiction problems **under uncertainty, …**

- Extensions to **Defender-Attacker-Defender (DAD)** Models (**trilevel games**)

- **Benders-like decomposition for general mixed-integer bilevel optimization**

THANK YOU FOR YOUR ATTENTION!

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  SOLVER: https://msinnl.github.io/pages/bilevel.html

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