A Node-Based ILP Formulation for the Node-Weighted Dominating Steiner Problem

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October 6, 2015

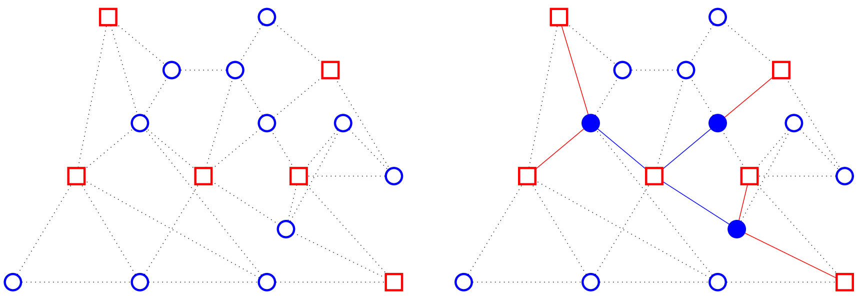
Abstract

In this article we consider the Node-Weighted Dominating Steiner Problem. Given a graph with node weights and a set of terminal nodes, the goal is to find a connected node-induced subgraph of minimum weight, such that each terminal node is contained in or adjacent to some node in the chosen subgraph. The problem arises in applications in the design of telecommunication networks.

Integer programming formulations for Steiner problems usually employ a variable for each edge. We introduce a formulation that only uses node variables and that models connectivity through node-cut inequalities, which can be separated in polynomial time. We discuss necessary and sufficient conditions for the model inequalities to define facets and we introduce a class of lifted partition-based inequalities, which can be used to strengthen the linear relaxation. Finally, we show that the polyhedron defined by these inequalities is integral if the underlying graph is a cycle where no two terminals are adjacent. In the general cycle setting, we show that we can get a complete description of the feasible solutions by lifting and projecting into a polytope with no more than twice the dimension. Finally, we show that the well-known indegree equalities are implied by the lifted partition inequalities.

1 Introduction

We consider the Node-Weighted Dominating Steiner Problem, denoted by NWDSTP, which is defined as follows: Given a graph $G = (V, E)$, a set $T \subset V, |T| \geq 2$ of terminal nodes, and a weight function $c : V \to \mathbb{R}$ on the nodes of G, we seek for a connected subgraph of minimum node weight such that each terminal is contained in the chosen subgraph or adjacent to a node in the subgraph. We denote $n = |V|$, $m = |E|$ and $k = |T|$. In graph theory, a node is said to *dominate* itself and all its neighbors. Hence, NWDSTP seeks for a least expensive connected subset of V that dominates T. We will also refer to a solution of NWDSTP as a connected T-dominating subset. Note that, due to the nature of the objective function, we are only interested in the node set of the optimal solution. In terms of the characteristics of this node set, it is sufficient to look for a subgraph induced by these nodes that satisfies the following side constraints: (1) the induced subgraph must be connected and (2) it has to dominate the set of terminals. Figure 1 illustrates an input graph and its feasible solution.



(a) Input graph. (b) A feasible solution.

Figure 1: An exemplary instance. Terminal nodes are shown as squares. Filled circles are the nodes selected to be part of the T-dominating set.

This problem arises in several practical applications related to the design of telecommunication or logistics networks. Such networks typically consist of two (or more) administrative or technology levels. Nodes of the higher level typically represent a core node or a hub and must be equipped with a higher-level technology or provide some features which require additional setup costs. Customer traffic enters the network at the lower access network level and then is sent to a core node, where it is aggregated with other customers' traffic to be transported through the core network more efficiently. After traversing the core network, the traffic is disaggregated and sent again within the access network level to its destination. In many applications, transportation within the access network is restricted to use direct connections between customers and core/hub nodes for technological or administrative reasons. In situations, where the overall network costs consist only or are strongly dominated by the costs of setting up core/hub nodes, the task of finding a minimum cost network naturally leads to the node-weighted dominating Steiner problem. In the area or telecommunications, this is the case in the planning of virtual networks and virtual function placement in the context of cloud services or in the planning of optical overbuilds for existing copper-based access networks, for example. Customer nodes represent the set of terminals T, and the node subset corresponding to the optimal NWDSTP solution provides the optimal location for core/hub nodes.

There are several related problems, which have been studied in the literature. In the Group Steiner Tree Problem, we are given several groups of nodes and seek for a tree that contains at least one node from each group. One easily verifies that the node weighted variant of this problem, where we seek for a tree that minimizes the weight of the nodes contained in the tree, is polynomially equivalent to NWDSTP. Any given instance of NWDSTP can be transformed into an equivalent instance of the Group Steiner Tree Problem by replacing each terminal node t by the terminal group consisting of t and all its neighbors. Reversely, an instance of of the Group Steiner Tree Problem can be reduced to an NWDSTP instance which is obtained by adding for each terminal group T_i one additional node i connected to all nodes in T_i to the graph, replacing the requirement to contain one node from the terminal group T_i by the requirement to dominate this individual node i, and setting the cost of the added nodes to a value that is larger than the sum of the costs of all original nodes. This ensures that no optimal solution will actually contain any of the added nodes. Instead, they will be dominated by one of their neighbors from the given terminal group and, hence, the solution contains a Group Steiner tree within the original graph. Garg, Konjevod, and Ravi [11] gave a polylogarithmic approximation algorithm for the problem. Demaine, Hajiaghayi, and Klein [6] improved the ratio when the graph is planar embedded and each group is the set of nodes on a face. A fault-tolerant version of the problem is considered by Khandekar, Kortsarz, and Nutov [17]. Lower bounds for the approximability of the problem are studied by Halperin and Krauthgamer [15].

Another problem closely related to NWDSTP is the Minimum Dominating Set Problem (MDS). In this problem, the goal is to find a minimum-cost vertex set that dominates the entire graph. The problem is one of the classic NP-complete problems considered by Garey and Johnson [10]. The MDS problem as well as its connected version, where the dominating set must induce a connected subgraph, are shown to be NP-complete. Furthermore, Bar-Yehuda and Moran [2] showed that the MDS problem is polynomially equivalent to the set cover problem. Thus, the strong logarithmic inapproximability threshold for the set cover problem shown by Feige [7] carries over in a straightforward way to the MDS problem. Finally, Hedetniemi, Laskar, and Pfaff [16] proposed a linear-time algorithm for the special case where G is a cactus graph.

Minimum Connected Dominating Set Problem (MCDS) is the problem in which we seek for a least cost subset of nodes that dominates the whole graph. Recently, Gendron et al. [12] proposed a branch-and-cut algorithm, a Benders decomposition approach, and a hybridization of the latter two for solving the problem to optimality. Strong logarithmic inapproximability bounds hold for the MCDS as well. Guha and Khuller [13] presented a $(3 \ln n)$-approximation algorithm for the the node-weighted MCDS, which they improved in [14] to a $((1.35+\varepsilon)\ln k)$-approximation algorithm. A $(\ln\delta+2)$-approximation algorithm, where δ denotes the maximum degree in the graph, is presented in [20]. For a more comprehensive literature overview on the MCDS and its relation to the Maximum Leaf Spanning Tree Problem (which unfortunately does not carry over to the NWDSTP), see [12].

Finally, the classical Steiner Tree Problem and its node-weighted variant are very closely to related to NWDSTP. In these problems we are given a graph $G = (V, E)$ and a subset $T \subset V$ of terminals and wish to find a tree of minimum weight that includes all terminals. The classical edge-weighted problem version is known to be NP-hard for many metrics. The node weighted version is considered by Klein and Ravi [18], where the authors developed $(2 \ln k)$-approximation algorithm and proved that the problem is as hard to approximate as the set cover problem. A $(1.5 \ln k)$-approximation was given in [14].

In this paper, we are interested in "thin" integer linear programming (ILP) formulations for NWDSTP using only $O(n)$ variables. Many well established ILP formulations for Steiner trees and related problems are based on undirected or directed edge variables and (multi-commodity) flow or connectivity constraints. Such models typically lead to a very large number of variables in the resulting formulations. However, for applications where costs arise only at the nodes or even only at the internal transit nodes contained in the solutions (as this is the case for NWDSTP), edge variables introduce an unnecessary modeling overhead that may harm the computational performance. We therefore propose a formulation that uses node variables only and that models connectivity through node-cut inequalities that can be separated in polynomial time. The resulting model contains a substantially smaller number of variables than the commonly used edge based models. Thus, our model is expected to lead to a better computational performance when solving very large problem instances using cutting plane approaches. We should mention that only very recently, node-based ILP models gained in popularity in modeling Steiner-trees and related problems. For example, node-based models for the (prize-collecting) Steiner tree were one of the most important ingredients of the implementation of Fischetti et al. [8], with which the authors managed to solve to provable optimality some of the long standing unsolved benchmark instances from the public libraries. Node-based models have also been used in forestry applications [3] and in bioinformatics [1]. Finally, a polyhedral study for the related connected subgraph polytope based on node-variables is given in [21].

The remainder of this paper is organized as follows. In Section 2 we introduce the basic notation used in this paper, our node variable based integer linear programming formulation of NWDSTP, and the polyhedron P defined by all feasible solutions. The fundamental properties of feasible solutions and the NWDSTP polyhedron P are then discussed in Section 3. In Section 4 we study under which conditions the original model inequalities define facets of P. In Section 5, we introduce and analyze partition inequalities based on node-separators, which can be added to the original model in order to strengthen its linear relaxation. We show that these inequalities are valid for P and in fact do strengthen the linear relaxation of the model. For the special case where the underlying graph is a cycle and no two terminals are adjacent, we prove that a formulation containing all partition inequalities yields an exact description of P, that is, already the linear programming relaxation of such a formulation yields integer optimal solutions. This result generalizes to cactus graphs in a straightforward way. In Section 6 we draw our conclusions.

2 Integer programming formulations

Before we can formally define the model considered in this paper, we need to introduce some basic notation.

For a node set $U \subseteq V$, we denote by $G[U]$ the subgraph induced by U. A subset $S \subset V$ is called a *separator* (of G) if $G[V \setminus S]$ is disconnected. A node subset $S \subset V$ is called *k, ℓ-separator* if $k, \ell \in V \setminus S$ and the nodes k and ℓ lie in different components of $G[V \setminus S]$. The set of all k, ℓ-separators is denoted by $\mathcal{S}_{k\ell}$. For notational simplicity, we typically write $G - S$ for $G[V \setminus S]$. A separator or a k, ℓ-separator S is called *minimal* if no proper subset $S' \subsetneq S$ is a separator or k, ℓ-separator, respectively. Given a separator S and $v \in V \setminus S$, we denote by $C_v \subseteq V$ the (nodes of the) connected component of $G - S$ that contains v.

For a node $v \in V$, we denote by Γ_v^* the set of all nodes adjacent to v. Furthermore, we let

$$\Gamma_v := \Gamma_v^* \cup \{v\}$$

be the set of all neighbors of v and v itself.

Now we are ready to introduce an integer programming formulation for NWDSTP. Our model uses only the binary node variables $y_v \in \{0, 1\}$, $v \in V$.

These are interpreted as

$$y_v = \begin{cases} 1 & \text{if } v \text{ is contained in the dominating Steiner tree} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the set $I^y := \{v \in V \mid y_v = 1\}$ of nodes contained in the chosen dominating Steiner tree is not required to contain all terminals in T, but only to be a dominating set for T. Given a vector $y \in \mathbb{R}^V$ and a set $S \subseteq V$, we typically write $y(S) := \sum_{v \in S} y_v$ in order to simplify notation.

Using these variables, we can formulate NWDSTP as follows:

$$\text{(NWDSTP):} \quad \min \sum_{v \in V} c_v y_v$$

$$\begin{aligned} y(S) &\geq y_k + y_\ell - 1 & \forall k, \ell \in V \setminus T,\, k \neq \ell,\, S \in \mathcal{S}_{k\ell} \quad (1) \\ y(S) &\geq y_k & \forall k \in V \setminus T,\, \ell \in T,\, S \in \mathcal{S}_{k\ell} \quad (2) \\ y(S) &\geq 1 & \forall k, l \in T,\, S \in \mathcal{S}_{kl} \quad (3) \\ y(\Gamma_v) &\geq 1 & \forall v \in T \quad (4) \\ y_v &\in \{0,1\} & \forall v \in V \end{aligned}$$

One easily verifies that (NWDSTP) is a correct integer linear programming model for the node-weighted dominating Steiner tree problem. Clearly, all inequalities (1)–(4) are valid for all incidence vectors of T-dominating Steiner trees and the objective function properly models the node weight. To see that the given constraints are sufficient, let $I = I^y := \{v \in V \mid y_v = 1\}$ be the set of nodes defined by a solution of (NWDSTP). Inequalities (3) ensure that for each terminal node pair $k, \ell \in T$ and each k, ℓ-separator S with k and ℓ in different components of $G - S$ at least one node from S is contained in I. This implies that I intersects each terminal separator. So, for all $k, \ell \in T$ that are not direct neighbors, I contains a neighbor of k, a neighbor of ℓ, and both are connected within I. Similarly, inequalities (1) and (2) require that I intersects each k, ℓ-separator for any node pair $k, \ell \in T \cup I$. Together with inequalities (3) this implies that the chosen nodes I induce a connected subgraph of G. Finally, inequalities (4) imply that each terminal node $i \in T$ is itself contained or has a neighbor in I. Thus, I forms a connected set dominating T, a dominating Steiner tree. Figure 2 illustrates infeasible cases that may occur by leaving out some of the constraints (1)–(4).

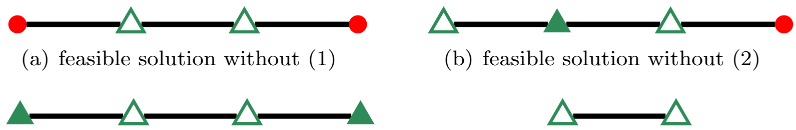
The convex hull of all integer solutions of (NWDSTP) defines the polyhedron

$$P := \text{conv}\{y \in \{0,1\}^V \mid y \text{ satisfies (1)–(4) }\}.$$

Clearly, P is nothing but the convex hull of the characteristic vectors of all connected sets $I \subseteq V$ that dominate the terminal set T. We call P the NWDSTP polyhedron.

3 Basic properties

In the following section, we will study which of the constraints (1)–(4) define facets of polyhedron P. Prior to that, we discuss some of the basic properties of P and of separators.



(a) feasible solution without (1)
(b) feasible solution without (2)
(c) feasible solution without (3)
(d) feasible solution without (4)

Figure 2: Necessity of model inequalities. Red circles are Steiner nodes taken into “solution”, green triangles are terminals, filled nodes are terminals that are part of the “solution”.

Lemma 3.1. *If G is 2-connected, then* $\dim(P) = n$.

Proof. Obviously, we have $\dim(P) \leq n$. To see that $dim(P) \geq n$, we now construct $n+1$ affinely independent vectors in P.

Let $v \in V$ be arbitrary. As G is 2-connected, $G - v$ is connected. Clearly, all nodes in V are either contained in $G - v$ or adjacent to a node in $G - v$. Thus, for each $v \in V$, the vector x^v with

$$x^v_u := \begin{cases} 1 & v \neq u \\ 0 & v = u \end{cases}, \quad u \in V$$

is a feasible solution of (NWDSTP) and hence contained in P. The same trivially holds for the vector $\mathbf{1}_n$ of all-ones. Since $\mathbf{1}_n - x^v = \mathbf{e}_v$, the $n+1$ vectors $\mathbf{1}_n$ and x^v, $v \in V$, are affinely independent. Thus $dim(P) \geq n$. □

We assume throughout the remainder of this paper that G is 2-connected. This can be done without loss of generality. If the underlying graph is not 2-connected, we can easily decompose the problem on the overall graph into the corresponding problems on its 2-connected components, the blocks, and consider each of them separately.

Next, assume we are given a separator or a k,ℓ-separator S that is not minimal and consider the corresponding separator inequality (1)-(3). As S is not minimal, there exists a smaller (in fact, even a minimal) separator or k,ℓ-separator $S' \subsetneq S$. Obviously, the corresponding separator inequality (1)-(3) for S' dominates the one for S: The inequality for S can be obtained by the inequality for S' by adding the non-negativity constrains for all $v \in S \setminus S'$. Hence, only minimal separators can induce facet-defining inequalities of P.

In order to characterize which minimal separators actually do induce facets of P, we need some further properties.

Lemma 3.2. *Let G be 2-connected, S be a minimal separator and $v \in S$. Then there is a spanning tree B on G such that v is the only inner (i.e. non-leaf) node of B that is contained in S.*

Proof. As S is minimal, $G' := G - (S \setminus \{v\})$ is connected, while $G - S$ is not. Let B' be a spanning tree in G'. Since $G - S$ is not connected, v cannot be a leaf node of B' but must be an inner node. Otherwise $B' - v$ would be a spanning tree of $G - S$.

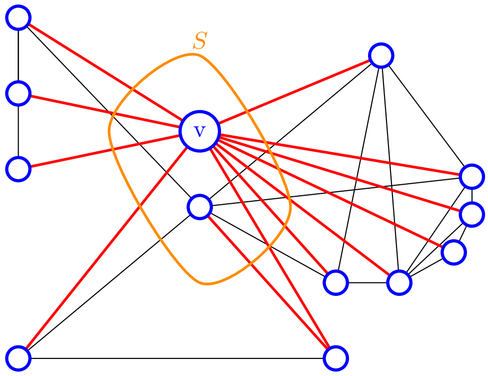


Figure 3: Illustration for 3.2: A possible tree B' is shown in red

As S is a minimal separator, each $i \in S \setminus \{v\}$ is contained in an edge leaving S. Now pick one such edge for each $i \in S \setminus \{v\}$ and denote it by e_i. Finally, set

$$B := B' \cup \bigcup_{i \in S \setminus \{v\}} e_i.$$

With this construction, every node $i \in S \setminus \{v\}$ has degree 1 with respect to B and, as B' spans $G - (S \setminus \{v\})$, B spans G. Hence, B has the claimed properties. □

Finally, we observe that *each* node in a minimal separator S contains edges to *all* connected components of $G - S$.

Lemma 3.3. *Let S be a minimal separator and let $C_1, \ldots, C_k$ be the connected components of $G - S$. Then, for each $v \in S$ and each $j \in \{1, \ldots, k\}$, there is an edge uv with $u \in C_j$.*

Proof. Assume the claim was wrong. Then there is a node $v \in S$ and a component C_j of $G - S$ such that $uv \notin E$ for all $u \in C_j$. Let $w \in C_j$. As C_j is a component of $G - S$ with $w \in C_j$, $v \notin C_j$, and $uv \notin E$ for all $u \in C_j$, the set $S' := S \setminus \{v\}$ is a w, v-separator. This contradicts the minimality of S. □

4 Investigation of the model inequalities

We now investigate under which conditions the model inequalities (3) and (4) define facets of P. Note that inequalities of type (2) and (1) are lifted variants of (3). They are obtained for the cases where only one or none of the components of $G - S$ contain a terminal nodes in a straightforward way via lifting the variables y_k or the variables y_k and y_ℓ that occur in the right hand side of the corresponding inequality from value 1 to value 0. Conditions for these inequalities to be facet defining can be derived from those for inequality (3) for the same separator S and terminal set $T' = T \cup \{k\}$ or $T' = T \cup \{k, \ell\}$, respectively.

In the case that every node in G is a terminal, conditions for (3) to be facet-defining were given in Theorem 3.9 of Fujie [9]. We now discuss the conditions for the general case.

Recall that, given a separator S and $v \in V \setminus S$, C_v denotes the connected component of $G - S$ that contains v. We call a node $v \in V \setminus S$ *S-replaceable* if there exists a node $j \in S$ such that the subgraph $G[(C_v \setminus \{v\}) \cup \{j\}]$ induced by nodes of C_v without v but with j instead is connected. In other words, the connected component C_v remains connected, if we replace node $v \in C_v$ by $j \in S$.

Theorem 4.1. *Let G be 2-connected and S be a minimal separator. If each $v \in V$ is S-replaceable, then (3) is a facet of P.*

Proof. Let $F_S := \{y \in P \mid y(S) = 1\}$ be the face of P that is induced by the inequality (3) for S. Assume that F_S is contained in a facet F induced by some valid inequality $\sum_{i \in V} \alpha_i y_i \geq \alpha_0$ for P, i.e.,

$$F_S \subseteq F := \left\{ \sum_{i \in V} \alpha_i y_i = \alpha_0 \right\}.$$

We will show that this implies

$$\alpha_v = \left\{ \begin{array}{cc} \alpha_0 & v \in S \\ 0 & v \notin S \end{array} \right. ,$$

which, in turn, implies that $F_S = F$ and F_S is a facet.

Our proof consists of two parts: First, we show that $\alpha_v = 0$ for all $v \notin S$. Then, in the second part, we show that for all $v \in S$ we have $\alpha_v = \alpha_0$. Throughout this proof we will often interpret binary vectors $y \in \{0,1\}^V$ as node sets $I^y := \{v \in V \mid y_v = 1\} \subseteq V$ and a node sets $I \subseteq V$ as their characteristic vectors $y^I = \chi^I \in \{0,1\}^V$. Given a binary vector $y \in P$, the set I^y then corresponds to the chosen connected dominating set. For a given binary vector y that is not necessarily in P, we say that y or its node set I^y dominates all nodes in the neighborhood of I, i.e., all nodes in $\{v \in V \mid v \in I \text{ or } uv \in E \text{ for some } u \in I\}$.

Clearly, for any spanning tree $B \subseteq E$ of G, the set $I(B) \subseteq V$ of *inner* (non-leaf) nodes of B defines a connected dominating set. Using this observation, we will now construct different spanning trees in G, whose inner node sets define vectors proving our claims.

To show the first claim, namely that $\alpha_v = 0$ for all $v \notin S$, it suffices to construct a spanning tree that has exactly one inner node in S and node v as a leaf. So, let $v \in V \setminus S$. We denote the node sets of the connected components of $G - S$ by $C_1, \ldots, C_r$. Without loss of generality we may assume $v \in C_1$. Let $j \in S$ be a replacement node for v, i.e., a node $j \in S$ such that $G[(C_v \cup \{j\}) \setminus \{v\}]$ is connected. Let $M := |S| - 1$.

We choose an arbitrary spanning tree $B_1 = (V_1, E_1)$ within $G[C_1 \cup \{j\} - \{v\}]$ and arbitrary spanning trees $B_2 = (V_2, E_2), \ldots, B_r = (V_r, E_r)$ within the connected components $C_2, \ldots, C_r$, respectively. Lemma 3.3 implies that there exists an edge from j to each component C_k, $k = 2, \ldots, r$, because S is minimal. For each component C_k, $k = 2, \ldots, r$, we choose one of these edges and denote it by c_k.

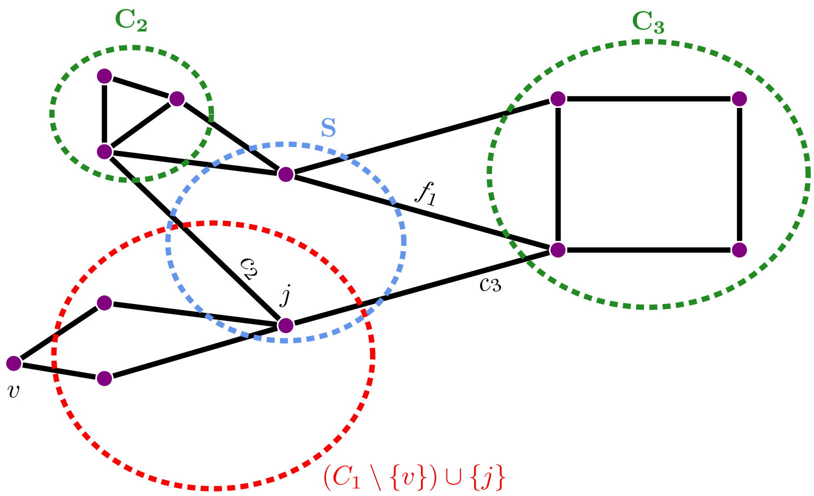


Figure 4: Illustration for proof of Theorem 4.1

Next, we choose edges $f_2, \ldots, f_M$ such that each $n \in S \setminus \{v\}$ is part of at least one edge to some component C_i. Finally, we set

$$E' := \bigcup_{i=1}^{r} E_i \cup \{c_i \mid i = 1 \ldots r\} \cup \{f_i \mid i = 2 \ldots M\}.$$

Figure 4 illustrates this construction.

One easily verifies that E' is a tree that it spans all nodes in V except for v. Hence, the set $I = I(E') \cup \{V_1 \setminus v\}$ consisting of all inner nodes $I(E')$ of E' plus all nodes in C_1 except v is a connected dominating set. Consequently, its characteristic vector y^I belongs to P. As all nodes in $S \setminus \{j\}$ are leaf nodes in E', y^I also satisfies the equality $y^I(S) = 1$, and hence $y^I \in F_S$.

As v is adjacent to a node in I, also $J := I \cup \{v\}$ is a connected dominating set and its characteristic vector y^J satisfies $y^J \in P$ and $y^J(S) = 1$.

Consequently, both vectors y^I and y^J are contained in $F_S \subseteq F$, which implies

$$\sum_{i \in V} \alpha_i (y_i^I - y_i^J) = \alpha_v = 0 \ .$$

This concludes the first part of the proof.

To prove the second claim, namely $\alpha_v = \alpha_0$ for all $v \in S$, let $v \in S$. Note that S is inclusion-wise minimal. Thus, Lemma 3.2 implies that there exists a spanning tree B of G such that v is the only inner node of B within S. Let y^I be the characteristic vector of the inner nodes $I := I(B)$ of B. Clearly, I defines a connected dominating set, so we have $y^I \in P$. Since there is only one inner node within S, we have $\sum_{i \in S} y_i^I = 1$, which implies $y^I \in F_S \subseteq F$. Finally, $I \cap S = \{v\}$ implies $\sum_{i \in S} \alpha_i y_i^I = \alpha_v = \alpha_0$, which concludes the proof of the second claim.

Consequently, any inequality inducing F has the form $\sum_{i \in S} \alpha_0 y_i = \alpha_0$ and, hence, $F_S = F$ is a facet. □

Next, we study the inequalities of type (4).

Consider inequality (4) for some terminal node $r \in T$ and assume there exists another terminal node $j \in V \setminus \Gamma_r$. If Γ_r^* is not a minimal separator, then inequality (4) for r is trivially dominated by the separator inequality (3) for any minimal separator S which is a (proper) subset of Γ_r^* and, hence, (4) cannot define a facet of P. If, on the other hand, Γ_r^* is a minimal separator, a sufficient condition for inequality (3) and thus also for (4) to be facet-defining is given in Theorem 4.1.

In the following, we thus assume that $V \setminus \Gamma_r$ does not contain any terminal. This case may occur if the set of terminals forms a highly connected cluster in the underlaying graph, for example. For this special case, a complete characterization of the cases when inequalities of type (4) define facets of P is given by the the following two theorems. In the first theorem, we define the conditions that must be satisfied and prove that these are sufficient for inequality (4) to be facet-defining. The necessity of the constraints is then shown in the second theorem.

Theorem 4.2. *Let G be 2-connected, $r \in T$, and $T \subseteq \Gamma_r$. For each node $j \in V \setminus \Gamma_r$, let C_j denote the component of $G - \Gamma_r$ that contains node j. Then inequality $y(\Gamma_r) \geq 1$ defines a facet of P if both conditions (C1) and (C2) hold:*

(C1) For all $j \in V \setminus \Gamma_r$, the graph G contains a tree B that

- *(i) spans all terminals,*
- *(ii) contains at least one node from the component C_j,*
- *(iii) does not use j as an inner node, and*
- *(iv) uses exactly one node from Γ_r as an inner node.*

(C2) For all $j \in \Gamma_r$, the graph G contains a tree B that

- *(i) spans all terminals,*
- *(ii) contains j, and*
- *(iii) uses j and only j as an inner node in Γ_r.*

Note that condition (C1-ii) is equivalent to the condition that node j itself is used a as a leaf-node in the tree B.

Proof. Let $F_r = \{y \in P \mid y(\Gamma_r) = 1\}$ be the face of P induced by the inequality $y(\Gamma_r) \geq 1$. Let furthermore F be a facet of P containing F_r and assume that F is induced by the inequality $\sum_{i \in V} \alpha_i y_i \geq \alpha_0$, i.e., $F_r \subseteq F = \left\{y \in P \mid \sum_{i \in V} \alpha_i y_i = \alpha_0\right\}$. We show that the two conditions (C1) and (C2) imply

$$\alpha_j = \begin{cases} \alpha_0 & j \in \Gamma_r \\ 0 & j \notin \Gamma_r \end{cases},$$

which, in turn, implies that inequality $\sum \alpha_i y_i \geq \alpha_0$ is a multiple of inequality $y(\Gamma_r) \geq 1$. As P is bounded and full-dimensional, it then follows that $F_r = F$ is a facet.

The proof consists of two parts: In the first part, we show that $\alpha_j = 0$ for $j \in V \setminus \Gamma_r$. Then, in the second part, we show that $\alpha_j = \alpha_0$ for each $j \in \Gamma_r$.

For the first part, we let $j \in V \setminus \Gamma_r$ and choose a tree $B = (V_j, E_j)$ that satisfies condition (C1).

With (C1-i), we clearly have $T \subseteq V_j$. Due to (C1-iii), B does not use j as an inner node. If $j \in B$, then node j is a leaf of B. Otherwise, we need to modify the tree B. Let $R = (r_1, f_1, r_2, f_2, \dots, f_{k-1}, r_k = j)$ be a shortest path within component C_j that connects some arbitrary node of B to j, where r_i denote the nodes and f_i denote the edges of R. Such a path exists, because (C1-ii) implies that B contains at least one node from C_j, and C_j is a connected component. As we chose a shortest path, $r_1 \in B$ and $r_2, \dots, r_k \notin B$. Now we add the path R to B. Clearly, this yields a new tree B that contains j as a leaf node. Also note that we did not add any node from Γ_r to B.

Next, let $I = I(B)$ be the set of all internal nodes of B. Because B spans all terminals, I is a connected dominating set for T and its characteristic vector y^I defines a feasible point in the polyhedron P. Because B uses exactly one node in Γ_r as an inner node, we also have $|\Gamma_r \cap I| = 1$. Thus, $y^I \in F_r \subseteq F$.

Note that $y^I_j = 0$, because j was a leaf node in the tree B. Thus, j is adjacent to some node in I and $J := I \cup \{j\}$ defines a connected dominating set for T as well. Consequently, also the characteristic vector y^J is contained in P. As $j \notin \Gamma_r$, y^J also satisfies $y^J(\Gamma_r) = 1$ and, consequently, also $\sum_{i \in V} \alpha_i y^J_i \geq \alpha_0$.

Subtracting the two equalities for y^I and y^J, we obtain our first claim

$$\sum_{i \in V} \alpha_i (y^J_i - y^I_i) = \alpha_j = 0.$$

To prove the second claim, let $j \in \Gamma^*_r$. By condition $(C2)$, there exists a tree B spanning j and all terminals, whose only inner node in Γ_r is j. We choose one such tree B and let $I := I(B)$ again be the set of its inner nodes. As B spans all terminals, I is a connected dominating set for T and its characteristic vector y^I is contained in P. As j is the only inner node of B within Γ_r, we also have $y^I \in F_r \subseteq F$, which implies our second claim $\alpha_j = \alpha_0$ and concludes the proof. □

We now show that the conditions (C1) and (C2) of Theorem 4.2 are also necessary.

Theorem 4.3. *Let G be 2-connected, $r \in T$, and $T \subseteq \Gamma_r$. If the inequality $y(\Gamma_r) \geq 1$ defines a facet of P, then both conditions (C1) and (C2) must hold.*

Proof. Assume that there exists a node j that violates either condition (C1) or condition (C2), depending on whether $j \in \Gamma_r$ or not. We show that then the stronger inequality

$$y(\Gamma_r) \geq 1 + y_j \tag{5}$$

holds for all $y \in P$. As the polyhedron P is full-dimensional and inequality (5) clearly dominates $y(\Gamma_r) \geq 1$, this implies that the latter inequality is not a facet of P.

Since P is defined as the convex hull of all integer solution of (NWDSTP), is suffices to show that (5) holds for all integer solutions y of (NWDSTP), that is, for all characteristic vectors of connected dominating sets.

Clearly, (5) holds for all integer solutions $y \in P$ with $y_j = 0$, for which it is equivalent to the model inequality $y(\Gamma_r) \geq 1$. So, we may restrict our attention to integer solutions $y \in P$ with $y_j = 1$.

First, we consider the case $j \in \Gamma_r$, which means that condition (C2) is violated for j. If there was a connected dominating set I with $j \in I$ and $I \cap \Gamma_r = j$, one easily obtains a tree B with internal nodes I that spans T and j and contains only j as an internal node within Γ_r by extending a spanning tree on I with edges from I to the terminals T. As (C2) is violated and, thus, such a tree does not exist, there exists no connected dominating set I with $j \in I$ and $I \cap \Gamma_r = j$. Hence, any connected dominating set I with $j \in I$ contains at least two nodes in Γ_r. Consequently, any integer solution $y \in P$ with $y_j = 1$ for $j \in \Gamma_r$ satisfies $y(\Gamma_r) \geq 2$ and, thus, (5).

Now, assume $j \in V \setminus \Gamma_r$, which means that condition (C1) is violated for j. Let $N := \{v \in V \mid uv \in E \text{ for some } u \in C_j\}$ be the set of nodes that are adjacent to some node in C_j. Note that any connected dominating set that also dominates the node j must contain at least one node of N. Otherwise N would separate j from r. Analogously to the previous case we now observe that there exists no connected dominating set I with $j \notin I$, $I \cap N \neq \emptyset$, and $|I \cap \Gamma_r| = 1$, because any such set I would result in a tree B that meets the requirements of constraint (C1). Hence, any connected dominating set I with $j \notin I$ and $I \cap N \neq \emptyset$ contains at least two nodes in Γ_r, implying that any integer solution $y \in P$ with $y_j = 1$ for $j \in V \setminus \Gamma_r$ satisfies $y(\Gamma_r) \geq 2 = 1 + y_j$.

Hence, if node j violates the corresponding constraint (C1) or (C2), then all integer solutions $y \in P$ satisfy (5). As discussed above, this conflicts with the assumtion that the inequality $y(\Gamma_r) \geq 1$ defines a facet of P. □

5 Partition inequalities

In this section, we show how the well-known Steiner partition inequalities studied by Chopra and Rao [4, 5] for edge-based formulations of the classical Steiner tree problem can be adapted to our setting. The classical edge-based Steiner partition inequalities are based on the observation that, given a partition of the node set of the graph, any feasible solution must contain at least as many edges between different components of the partition as are needed to connect all those components that contain terminals.

In order to apply this observation to our node-based setting, we consider separator node sets whose removal partitions the remaining graph into several components, instead of edge sets. Given a graph $G = (V, E)$ and a subset $S \subset V$, we denote by $C(S)$ the set of connected components of $G - S$. This set consists of the subsets

$$C(S) = T(S) \,\dot{\cup}\, N(S),$$

where $T(S)$ denotes the set of components containing terminals and $N(S)$ denotes the subset of components without terminals. For simplicity, we will refer to the components in $T(S)$ as *terminal components* and to those in $N(S)$ as *terminal-free components*. We also let $t_S := |T(S)|$ and $n_S := |N(S)|$.

Instead of simply counting the number of edges between different components, as in the classical edge-based Steiner partition inequalities, we must take care of the fact that the nodes in S may actually connect more than two components of $G - S$ and S in a solution. Clearly, any Steiner tree B that spans all terminals must contain at least $t_S - 1$ many edges between S and the different components of $G - S$ in order to connect all terminal components. Each node $v \in S$, however, is adjacent only to a certain number of components, and it provides connectivity to those components only if it is chosen to be in the solution. Furthermore, any Steiner tree B defines a forest within $G[S]$. Hence, the number of edges that B contains within each component of $G[S]$ is bounded by the size of this component minus one, and this bound reduces if some nodes of the component are not contained in the solution. The latter two observations allow us to bound the total contribution of a single node $v \in S$ to the overall connectivity of any Steiner tree B and motivate the following definitions.

Definition 5.1. *Let $G = (V, E)$, $S \subset V$, and $s \in S$. Denoting the connected component of s in $G[S]$ by C_s, we call*

$$q_S(s) := \frac{|C_s| - 1}{|C_s|}$$

the tree quotient *of the node s with respect to the separator S.*

Definition 5.2. *Let $G = (V, E)$, $S \subset V$, and $C(S) = T(S) \dot\cup N(S)$ as defined above. For $v \in S$, we define its S*-degree *as*

$$\delta_S(v) := |\{C \in C(S) \mid uv \in E \text{ for some } u \in C\}| + q_S(v)$$

Note that $\delta_S(v)$ is exactly the number of components of $G - S$ that v is adjacent to if v is not adjacent to any other node in S. Otherwise, if v also has neighbors in S, its S-degree is the number of adjacent components in $G - S$ plus a certain fraction pertaining to the number of edges it needs to use to build connections inside S.

Using this notation, we can formulate the basic Steiner partition inequalities for our node-based ILP model.

Theorem 5.3. *For each subset $S \subset V$, the following* Steiner partition inequality *is valid for P:*

$$\sum_{v \in S} (\delta_S(v) - 1)\, y_v \geq t_S - 1 \tag{6}$$

Proof. Consider any subset $S \subset V$ and a feasible solution y. We would like to show that y satifies inequality (6).

In any feasible solution y, the set $I^y := \{v \in V \mid y_v = 1\}$ forms a single connected component dominating all terminals. Let now B be a Steiner tree that spans the nodes of I^y and all terminal nodes. To simplify the calculations, we assume w.l.o.g. that any terminal component of $G - S$ only contains the terminal. Otherwise, we can contract these components to single terminals.

The tree B then contains at least $|T| + \sum_{i \in S} y_i$ many nodes.

We now bound the number of edges of B. Let $\delta_S^+(s)$ for $s \in S$ denote the out-degree of the node s with respect to S, that is, the number of edges whose other endpoint lies outside S. The number of edges in B between S and components of $G - S$ clearly is upper-bounded by

$$\sum_{i \in S} \delta_S^+(i) y_i.$$

Furthermore, a tree or a forest on n nodes can contain at most $n - 1$ edges. Denoting the connected components of $G[S]$ by $S_1, \ldots, S_k$, this implies that B

contains at most $|S_i|-1$ edges within each S_i. Hence, the total number of edges of B with both end-nodes in S cannot exceed

$$\sum_{i\in S} q_S(i)y_i.$$

As the number of nodes cannot be higher than the number of edges plus one, we get

$$\sum_{i\in S}(\delta_S^+(i)+q_S(i))y_i \geq -1+|T|+\sum_{i\in S} y_i \ ,$$

which implies

$$\sum_{i\in S}(\delta_S(i)-1)y_i \geq |T|-1 \ .$$

Thus, any $y \in P$ must satisfy the Steiner partition inequality (6). □

Note that, given a minimal k,ℓ-separator $S \in \mathcal{S}_{k,\ell}$ for arbitrary distinct terminals $k,\ell \in T$, its associated Steiner partition inequality (6) is implied by the minimal separator inequality (3): Due to minimality of S, Lemma 3.3 implies that all nodes in S are adjacent to all components in $C(S)$ and, hence, $(\delta_S(v)-1) \geq t_S - 1$. Multiplying (3) with $t_S - 1$, one obtains a stronger inequality than (6).

However, there are graphs and families of separators $S \subset V$ for which the Steiner partition inequalities are not implied by the inequalites of (NWDSTP) and where they actually strengthen the linear relaxation of this formulation.

Example 5.4. *Consider the example shown in Figure 5. The values shown next to the nodes correspond to the LP-relaxation values of the model* (1)-(4) *for the objective function $c_v = 1$ for all $v \in V$. Notice that each minimal separator is a subset $S' \subset V$ of cardinality two, hence all minimal separator inequalities are satisfied. Consider now an arbitrary subset S of three non-terminal nodes such that the number of terminal-components in $G-S$ is $t_S = 3$, for example $S = \{2,4,6\}$. Then the Steiner partition inequality* (6) *associated to S is violated by the depicted fractional solution.*

Increasing the length of the cycle graph in Example 5.4 to $2k$ nodes, one easily verifies that the ratio between the linear relaxation value obtained with the original formulation (NWDSTP) and the formulation with additional Steiner partition inequalities can be as large as $2-\frac{1}{k}$ for any $k > 2$. More precisely, the solution y' with $y'_v = 1/2$ for all $v \in V$ is an optimal solution for the LP-relaxation of (NWDSTP) for the objective function $c = 1$, with objective value equal to k. On the other hand, the solution $\tilde{y}$ with $\tilde{y}_v = (k-1)/k$ is an optimal LP-solution for the relaxation including the Steiner partition inequalities, with the objective value of $2(k-1)$.

In those cases where some components of $G-S$ contain no terminals, the corresponding Steiner partition inequality can be strengthened easily. Let $N(S) = \{N_1(S),\ldots,N_{n_S}(S)\}$, $N_i(S) \subset V$ be the set of terminal=free components of $G-S$ and consider ordered n_S-tuples $(v_1,\ldots,v_{n_S})$ of nodes such that $v_i \in N_i(S)$ for $i = 1,\ldots,n_S$. The set of all such n_S-tuples is denoted by $\mathcal{N}(S)$. Lifting the variables corresponding to one such n_S tuple into the Steiner partition inequality for S, we obtain a stronger inequality.

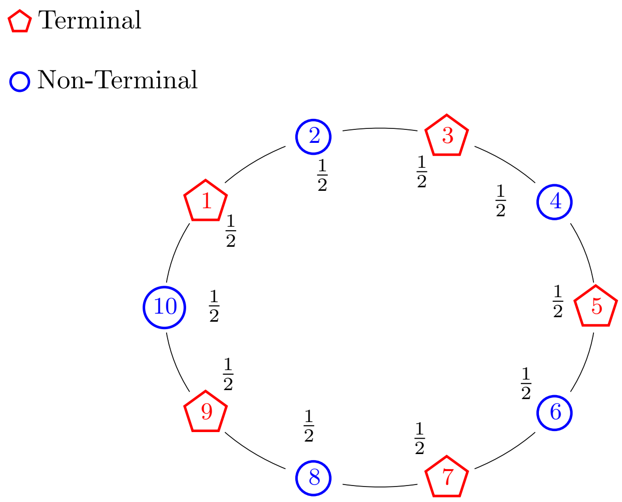


Figure 5: A valid solution for the NWDSTP relaxation (1)-(4) that is cut off by the Steiner partition inequalities (6).

Theorem 5.5. *For each subset $S \subset V$ and each tuple $(v_1, \dots, v_{n_S}) \in \mathcal{N}(S)$, the following* lifted Steiner partition inequality *is valid for P:*

$$\sum_{v \in S} (\delta_S(v) - 1)\, y_v - \sum_{i=1}^{n_S} y_{v_i} \geq t_S - 1 \tag{7}$$

Proof. The validity of lifted Steiner partition inequalities (7) can be shown analogously to the validity of the Steiner partition inequalities (6). Note that the tuple $(v_1, \dots, v_{n_S})$ contains one representative node for each terminal-free component $N_i(S)$ of $G - S$. If $y_{v_i} = 1$ for one such representative node v_i, then the component containing v_i needs to be connected to the rest of the solution. In this case, this component can be treated as if it contained a terminal. Consequently, the number of components that need to be connected by the solution y increases to t_S for the terminal-components plus $\sum_{i=1}^{n_S} y_{v_i}$ for the terminal-free components that contain chosen nodes. Hence, inequality (7) holds. □

There exist problem instances where the addition of lifted Steiner partition inequalities (7) strengthens the linear relaxation of (NWDSTP), even when compared to the relaxation obtained after adding the basic Steiner partition inequalities (6).

Example 5.6. *Consider the example shown in Figure 6. In the figure, we see an example of a valid solution for the NWDSTP relaxation. As one easily verifies, this solution is not cut away by the inequalities given in the (NWDSTP) formulation including Steiner partition inequalities* (7)*. Consider now the separator* $S = \{1, 3, 6, 9\}$*. The lifted partition inequality for this separator is*

$$y_1 + y_3 + y_6 + y_9 - y_0 - y_2 \geq 1,$$

which the given solution violates.

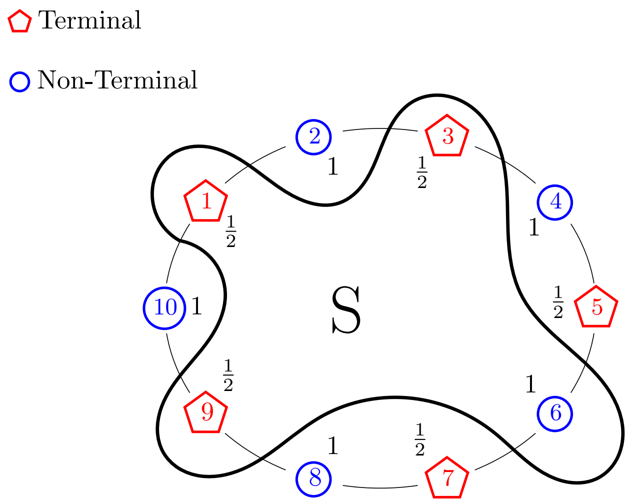


Figure 6: A valid solution for the NWDSTP relaxation The solution values are marked inside the circle, the separator $S = \{1, 3, 6, 9\}$ is depicted in black.

In the remainder of this section we show that in the special case where G is a cycle, the lifted Steiner partition inequalities (6) combined with the model constraints (1)-(4) are sufficient to describe the dominant of P.

5.1 A complete description of P on a cycle

Throughout this section we assume that $G = (V, E)$ is a cycle.

Definition 5.7. *Let $G = (V, E)$ be a cycle, $S \subset V$ and $C(S) = T(S) \dot{\cup} N(S)$ as defined above. For $v \in S$, we define its* circular S-degree *as*

$$\delta_S^C(v) := |\{C \in C(S) \mid uv \in E \text{ for some } u \in C\}| + \begin{cases} 1 & \text{if } \{u \in S \mid uv \in E\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}.$$

We switch to this weaker definition instead of the S-degree because this simplifies the proofs and yields the same results in cycle graphs.

We begin by studying the case where G consists of alternating terminal and non-terminal nodes. So, in the following let $V = \{0, \dots, 2k-1\}$, $E = \{\{i, i+1\} \mid i = 0, \dots, 2k-1\}$, and $T = \{v_0, v_2, \dots, v_{2k-2}\}$ for $k \geq 2$.

To simplify notation, we assume that the node index $2k$ is equivalent to the node index 0, or in other words, all calculations with node indices are to be understood as being executed in $\mathbb{Z}/2k\mathbb{Z}$.

We consider the linear programming relaxation of (NWDSTP) that is defined by the set of all lifted Steiner partition inequalities (7) and the variable boundary constraints only, i.e.,

$$\text{(PART*)}: \qquad \min \quad c^T y$$

$$\sum_{v \in S} y_v - \sum_{i=1}^{n_S} y_{n_i} \geq t_S - 1 \qquad \forall\, S \in \mathcal{S},\ (n_1, \ldots, n_{n_S}) \in \mathcal{N}(S)$$

$$1 \geq y_v \geq 0 \qquad \forall\, v \in V$$

It is easy to verify that all inequalities (1)-(4) of (NWDSTP) are implied by the inequalities of (PART*) if G is a cycle of alternating terminals and non-terminals. Hence, the polytope defined by the linear relaxation of (NWDSTP) is a relaxation of the polytope defined by (PART*). On the other hand, the polytope defined by (PART*) is a relaxation of P, because all lifted Steiner partition inequalities (7) are valid for P.

In the following, we show that the polytope defined by (PART*) is integral, that is, for each objective c there exists an integer optimal solution y^* for (PART*). This then directly implies that (PART*) is a complete description of P.

First, we observe that the lifted partition inequalities corresponding to non-stable sets S are redundant.

Lemma 5.8. *Let $S \in \mathcal{S}$ and $(n_1, \ldots, n_{n_S}) \in \mathcal{N}(S)$. If S is not a stable set, then inequality* (7) *is redundant in (PART*).*

Proof. Since S is a separator and G is a cycle, S consists of at least two connected components. If S is not a stable set, then at least one of these components contains more than one node. As G is a circle, this component is a path P in G. Without loss of generality we assume that the nodes are numbered in such a way that $P = (i, i+1, \ldots, i+\ell)$. Let $S' := S \setminus \{i, \ldots, i+\ell-1\}$. In words, S' is the separator obtained from S by replacing all nodes of P with the single node $i+\ell$.

Clearly, $G - S'$ contains exactly as many components as $G - S$. Furthermore, all components of $G - S$ that are not adjacent to i are also components of $G - S'$. Only the one component C of $G - S$ that is adjacent to i increases in $G - S'$ to a component $C' = C \cup \{i, \ldots, i+\ell-1\}$.

If C is a terminal component, then C' is a terminal component too. In this case, consider the lifted Steiner partition inequality (7) defined by S' and $(n_1, \ldots, n_{n_S})$. As the terminal-free of $G - S$ are exactly the terminal-free components of $G - S'$, we have $(n_1, \ldots, n_{n_S}) \in \mathcal{N}(S) = \mathcal{N}(S')$. Hence, the lifted partition inequality defined by S' and $(n_1, \ldots, n_{n_S})$ is valid and contained in the system (PART*). As $S' \subsetneq S$, the inequality for S' clearly dominates the one for S.

If C is a terminal-free component, then the larger C' must be a terminal component. In this case, we have $n_{S'} = n_S - 1$ and $t'_S = t_S + 1$. W.l.o.g. let C be the last terminal-free component of $G - S$, i.e., $n_{n_S} \in C$. As all other terminal-free components of $G - S$ remain terminal-free in $G - S'$, we have $(n_1, \ldots, n_{n_S - 1}) \in \mathcal{N}(S')$. Hence, the lifted Steiner partition inequality (7) defined by S' and $(n_1, \ldots, n_{n_S - 1})$ is valid and contained in the system (PART*). Together with the inequalities $y_{n_{n_S}} \leq 1$ and $y_v \geq 0$ for $v = i, \ldots, i+\ell-1$, this inequality implies the lifted Steiner partition inequality for S and $(n_1, \ldots, n_{n_S})$.

Consequently, inequality (7) is redundant if S contains a path. □

Due to Lemma 5.8, we may instead of PART* consider the formulation PART, which contains lifted Steiner partition inequalities only for stable sets S. This relaxation then defines the same polytope as PART*.

$$\text{(PART):} \qquad \min \quad c^T y$$

$$\sum_{v \in S} y_v - \sum_{i=1}^{n_S} y_{n_i} \geq t_S - 1 \quad \forall \text{stable } S \in \mathcal{S},\ (n_1, \ldots, n_{n_S}) \in \mathcal{N}(S)$$

$$1 \geq y_v \geq 0 \qquad \forall\, v \in V$$

In our next step, we show that, for any nonnegative objective function c, there exists an optimal solution for (PART) that satisfies the additional equalities

$$y_i = y_{i-1} + y_{i+1} - 1 \qquad \forall\, i \in T, \tag{8}$$

To facilitate this, we introduce the notion of an irreducible solution.

Definition 5.9. *A solution $y \in \mathbb{R}^V$ of (PART) is called* irreducible *if there is no solution $y' \in \mathbb{R}^V$ of (PART) with $y' \leq y$ and $y' \neq y$.*

Clearly, for any nonnegative objective function c, an optimal and irreducible solution y of (PART) exists. In the following two lemmas we show that irreducible solutions satisfy (8).

Lemma 5.10. *Let $c \in \mathbb{Q}^V_{\geq 0}$ be a nonnegative objective function and $y \in \mathbb{R}^V$ be an irreducible, optimal solution of (PART). Then we have*

$$y_i \geq y_{i-1} + y_{i+1} - 1 \qquad \forall\, i \in T\ . \tag{9}$$

Proof. Assume the claim was wrong. We have $y_i < y_{i-1} + y_{i+1} - 1$ for some $i \in T$. To simplify notation, let $u := i - 1$, $v := i$, and $w := i + 1$. Then

$$\varepsilon := y_u + y_w - 1 - y_v > 0\ . \tag{10}$$

We will show that $\bar{y} \in \mathbb{R}^V$ defined as

$$\bar{y}_i := \begin{cases} y_i - \frac{\varepsilon}{2} & \text{if } i \in \{u, w\} \\ y_i & \text{otherwise} \end{cases} \qquad \text{for } i \in V \tag{11}$$

satisfies (PART). As $\bar{y} \leq y$ and $\bar{y} \neq y$, this implies that y is not irreducible for (PART), which contradicts the preconditions of the lemma.

Obviously, $\bar{y}$ satisfies all boundary constraints $0 \leq y_i \leq 1$ of (PART). It remains to show that $\bar{y}$ also satisfies all non-redundant lifted Steiner partition inequalities (7).

So, let $S \in \mathcal{S}$, $(n_1, \ldots, n_{n_S}) \in \mathcal{N}(S)$, and consider the lifted Steiner partition inequality defined by S and $(n_1, \ldots, n_{n_S})$.

We now distinguish several cases, depending on which nodes belong to S.

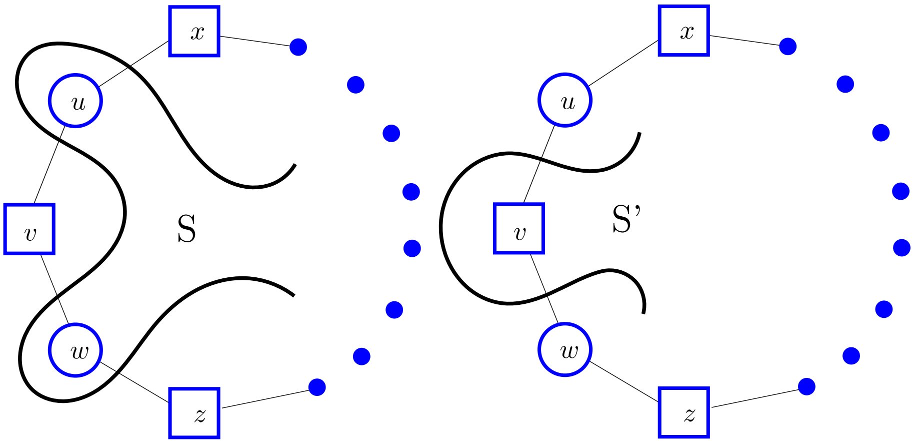


Figure 7: Lemma 5.10, Case 1

Case 1: $u, w \in S$

Due to Lemma 5.8, we may assume $v \notin S$. Otherwise the lifted Steiner partition inequality defined by S and $(n_1, \ldots, n_{n_S})$ is redundant (as S would then contain a path between u and w).

Let $S' := (S \setminus \{u, w\}) \cup \{v\}$, as illustrated in Figure 7.

If S' is not a separator, we must have $S = \{u, w\}$. In this case $G-S$ contains two terminal components (one is $\{v\}$, and the other contains all the remaining terminals $T \setminus \{v\}$). The (lifted) Steiner partition inequality defined by S then reads

$$y_u + y_w \geq 1 \ .$$

With (10) and (11), this inequality is trivially satisfied by $\bar{y}$.

If S' is a separator, the terminal-free components in $G - S'$ and $G - S$ are equal. Therefore, also the lifted Steiner partition inequality for S' and $(n_1, \ldots, n_{n_S}) \in \mathcal{N}(S) = \mathcal{N}(S')$ is part of the system (PART) and holds for y, i.e.,

$$\sum_{x \in S'} y_x - \sum_{i=1}^{n_S} y_{n_i} \geq t'_S - 1 \ .$$

On the other hand, the number $t_{S'}$ of terminal components in $G-S'$ is one less than the number t_S of terminal components in $G-S$, as the terminal component $\{v\}$ disappears. This implies

$$y_u + y_w - \varepsilon + \sum_{i \in S', i \neq v} y_i - \sum_{i=1}^{n_S} y_{n_i} \geq t_S - 1$$

and, together with (10) and (11),

$$\sum_{i \in S} \bar{y}_i - \sum_{i=1}^{n_S} \bar{y}_{n_i} \geq t_S - 1 \ .$$

Hence, $\bar{y}$ fulfills the lifted Steiner partition inequality for S and $(n_1, \ldots, n_{n_S})$.

Case 2: $u \notin S$, $w \notin S$

As u, w are not in S, the only way they can be involved in the lifted Steiner partition inequality is as one of the lifted non-terminal variables y_{n_i}. As these variables have a negative oefficient, however, the inequality trivially holds for $\bar{y}$ if it holds for y.

Case 3: $u \in S$, $w \notin S$ (or, analogously, $u \notin S$, $w \in S$)

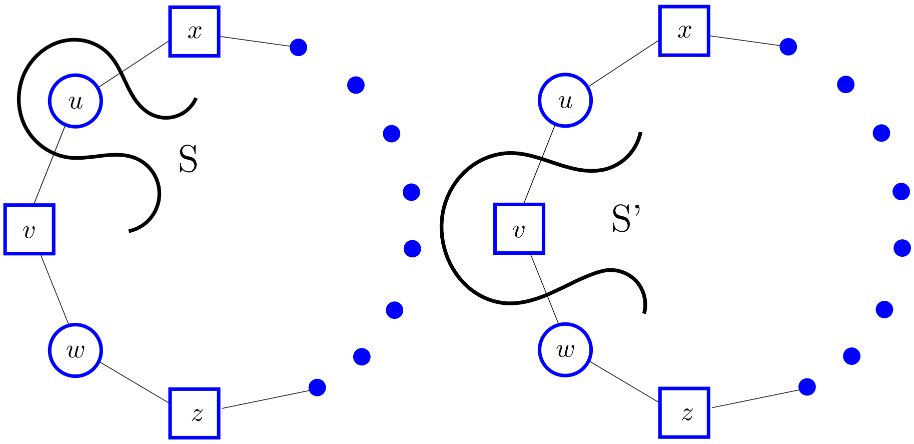


Figure 8: Lemma 5.10, Case 3

Let z be the neighbor of w which is not v and x be the neighbor of u which is not v, as illustrated in Figure 8. By Lemma 5.8 we may assume that $v \notin S$ and $x \notin S$.

We consider the separator $S' := (S \setminus \{u\}) \cup \{v\}$ obtained by replacing node u by node v in S. If S is a separator, then S' is as well, because $w \notin S$. Replacing u by v enlarges the component containing x, reduces the component containing w, and leaves all other components of $G - S$ unchanged. While the component containing x will be a terminal component in both $G-S$ and $G-S'$, the component containing w is a terminal component in $G-S$, but the reduced component in $G - S'$ might be a terminal-free component.

If $z \notin S$, then the component that contains w also contains the terminal x and is terminal component in both $G - S$ and $G - S'$. Thus, the number of terminal components and node sets of the terminal-free components are the same for $G-S$ and $G-S'$. In this case, also the lifted Steiner partition inequality for S' and $(n_1, \ldots, n_{n_S}) \in \mathcal{N}(S) = \mathcal{N}(S')$ is part of the system (PART) and holds for y, i.e.,

$$\sum_{i \in S'} y_i - \sum_{i=1}^{n_S} y_{n_i} \geq t'_S - 1 = t_S - 1 \ .$$

Adding (10) and $1 \leq y_w$ to this inequality, we obtain

$$\sum_{i \in S} y_i - \sum_{i=1}^{n_S} y_{n_i} \geq t_S - 1 + \varepsilon \ ,$$

and, with (11),

$$\sum_{i \in S} \bar{y}_i - \sum_{i=1}^{n_S} \bar{y}_{n_i} \geq t_S - 1 \; .$$

Otherwise, if $z \in S$, then the number t'_S of terminal components in $G - S'$ is one less than the number t_S of terminal components in $G - S$. Also, $G - S'$ contains the terminal-free component $\{w\}$, which has not been a terminal-free component in $G - S$. In this case, the lifted Steiner partition inequality for S' and $(n_1, \ldots, n_{n_S}, w) \in \mathcal{N}(S) \times \{w\} = \mathcal{N}(S')$, namely

$$\sum_{i \in S'} y_i - \sum_{i=1}^{n_S} y_{n_i} - y_w \geq t'_S - 1 = t_S - 2$$

is part of the system (PART) and holds for y. Together with (10) this implies

$$\sum_{i \in S'} y_i + y_u - y_v - \sum_{i=1}^{n'_S} y_{n_i} - y_w + y_w \geq t_S - 2 + 1 + \varepsilon \; ,$$

and with (11) finally

$$\sum_{i \in S} \bar{y}_i - \sum_{i=1}^{n_S} \bar{y}_{n_i} \geq t_S - 1 \; .$$

Hence, $\bar{y}$ fulfills the lifted Steiner partition inequality for S and $(n_1, \ldots, n_{n_S})$ also in this case, which concludes the proof. □

The validity of the reverse inequalities is shown in the following lemma.

Lemma 5.11. *Let $c \in \mathbb{Q}^V_{\geq 0}$ be a nonnegative objective function and $y \in \mathbb{R}^V$ be an irreducible optimal solution of (PART). Then we have*

$$y_i \leq y_{i-1} + y_{i+1} - 1 \qquad \forall\, i \in T \; . \tag{12}$$

Proof. Assume the claim was wrong and that there is some $i \in T$ with $y_i > y_{i-1} + y_{i+1} - 1$. Denoting again $u := i - 1$, $v := i$, and $w := i + 1$, this means

$$\varepsilon := y_v - y_u - y_w + 1 > 0 \; . \tag{13}$$

Similar to the proof of Lemma 5.10, we will show that $\bar{y} \in \mathbb{R}^V$ defined as

$$\bar{y}_i := \begin{cases} y_i - \varepsilon & \text{if } i = v \\ y_i & \text{otherwise} \end{cases} \qquad \text{for } i \in V \tag{14}$$

satisfies (PART), which cannot be the case if y is an irreducible solution of (PART).

Note that (PART) contains the lifted Steiner partition inequality $y_u + y_w \geq 1$ corresponding to $S = \{u, w\}$. With (13) this implies $1 \geq y_v - \varepsilon = y_u + y_w - 1 \geq 0$ and thus $1 \geq \bar{y}_v = y_v - \varepsilon \geq 0$. Hence, $\bar{y}$ satisfies all boundary constraints $0 \leq y_i \leq 1$ of (PART).

To show that $\bar{y}$ also satisfies all lifted Steiner partition inequalities (7) of (PART), let let $S \in \mathcal{S}$ and $(n_1, \ldots, n_{n_S}) \in \mathcal{N}(S)$ define a non-redundant lifted Steiner partition inequality. We may assume that S is a stable set, because otherwise the lifted Steiner partition inequality would be redundant.

If $v \notin S$, then the lifted Steiner partition inequality trivially holds for $\bar{y}$, because it holds for y and, as $v \in T$, the coefficient of variable y_v is 0. Hence, we may assume $v \in S$ for the rest of the proof.

Again, we distinguish several cases depending on S.

Case 1: $x, z \in S$

Let $S' := S - \{v\}$, as illustrated in Figure 9. Note that $u, w \notin S$, because S is assumed to be a stable set.

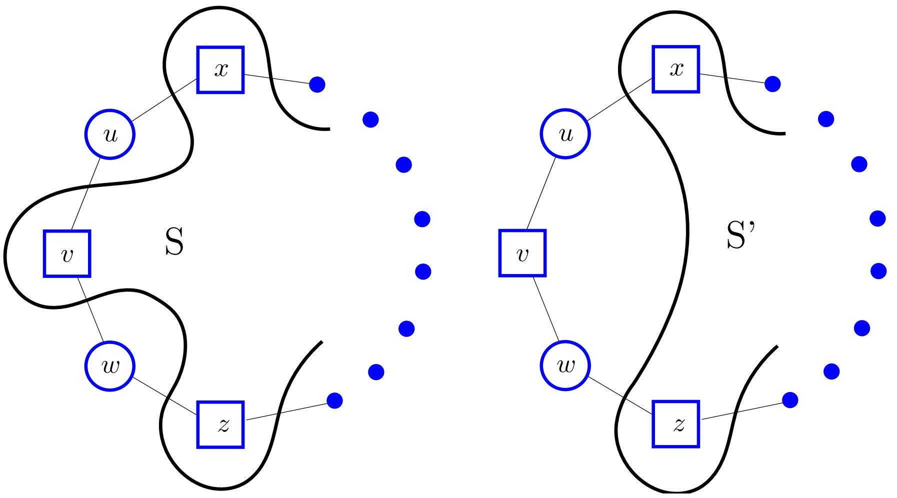


Figure 9: Lemma 5.11, Case 1

If S' is not a separator, we must have $x = z$, $S = \{x, v\}$, and $V = \{x, u, v, w\}$. Otherwise S would not be a stable set or not be a separator. In this case, the inequality belonging to S is

$$y_v + y_x - y_u - y_w \geq -1 \ .$$

which is trivially satisfied by $\bar{y}$.

If S' is a separator, then $G - S'$ contains one more terminal component than $G - S$, namely the component $\{u, v, w\}$, while the two terminal-free components $\{u\}$ and $\{w\}$ of $G - S$ are no longer components of $G - S'$. All other components of $G - S$ are also components of $G - S'$. As the two components $\{u\}$ and $\{w\}$ of $G - S$ contain only a single node each, the two non-terminal variables y_u and y_w necessarily occur in the lifted Steiner partition inequality for S and $(n_1, \ldots, n_{n_S})$ with a coefficient of -1. We may assume w.l.o.g. that the terminal-free components of $G - S$ are numbered in such a way that $u = n_{n_S - 1}$ and $w = n_{n_S}$. As all other terminal-free components remained unchanged, we have $\mathcal{N}(S) = \mathcal{N}(S') \times \{u\} \times \{w\}$. Thus, S' and $(n_1, \ldots, n_{n_S - 2}) \in \mathcal{N}(S')$ define the lifted Steiner partition inequality

$$\sum_{i \in S'} y_i - \sum_{i=1}^{n_{S'}} y_{n_i} \geq t_{S'} - 1 \ ,$$

which is part of the system (PART) and hence valid for y. Adding (13), we get

$$\sum_{i \in S'} y_i + y_v - \sum_{i=1}^{n_{S'}} y_{n_i} - y_u - y_w \geq t_{S'} - 2 + \varepsilon = t_S - 1 + \varepsilon \ .$$

With (14) this implies

$$\sum_{i \in S} \bar{y}_i - \sum_{i=1}^{n_S} \bar{y}_{n_i} \geq t_S - 1 \ .$$

In other words, $\bar{y}$ satisfies the lifted Steiner partition inequality for S and $(n_1, \ldots, n_{n_S})$.

Case 2: $z \in S$, $x \notin S$ (or, analogously, $z \notin S$, $x \in S$)

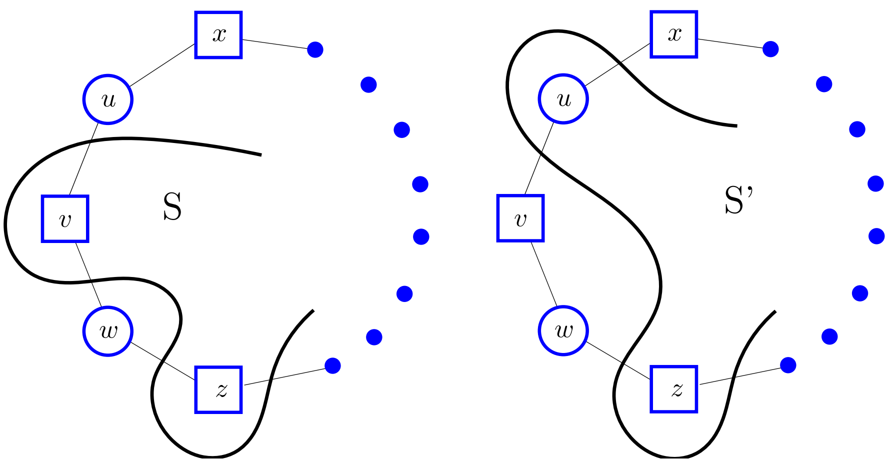


Figure 10: Lemma 5.11, Case 2

In this case, we let $S' := S \setminus \{v\} \cup \{u\}$, as shown in Figure 10. Since S is assumed to be a stable set, we have $u, w \notin S$.

Replacing v by u in the separator, the terminal-free component $\{w\}$ of $G - S$ becomes a terminal component $\{v, w\}$ in $G - S'$, and the terminal component containing x in $G - S$ becomes smaller in $G - S'$, as node u is removed from this component. All other components of $G - S$ remain unchanged in $G - S'$. As $\{w\}$ is a single node terminal-free component of $G - S$, the variable y_w must occur in the lifted Steiner partition inequality for S and $(n_1, \ldots, n_{n_S})$ with a coefficient of -1. Assuming that $w = n_{n_S}$ corresponds to the last terminal-free component in $N(S)$, we have $(n_1, \ldots, n_{n_S-1}) \in \mathcal{N}(S')$. Thus, lifted Steiner partition inequality for S' and $(n_1, \ldots, n_{n_S-1})$

$$\sum_{i \in S'} y_i - \sum_{i=1}^{n_S-1} y_{n_i} \geq t_{S'} - 1$$

is part of (PART) and holds for y. Adding (13), we get

$$\sum_{i \in S'} y_i + y_v - y_u - \sum_{i=1}^{n_S-1} y_{n_i} - y_w \geq t_{S'} - 2 + \varepsilon = t_S - 1 + \varepsilon \ .$$

With (14) this implies

$$\sum_{i \in S} \bar{y}_i - \sum_{i=1}^{n_S} \bar{y}_{n_i} \geq t_S - 1 \ ,$$

so the lifted Steiner partition inequality for S and $(n_1, \ldots, n_{n_S})$ holds for $\bar{y}$.

Case 3: $x, z \notin S$

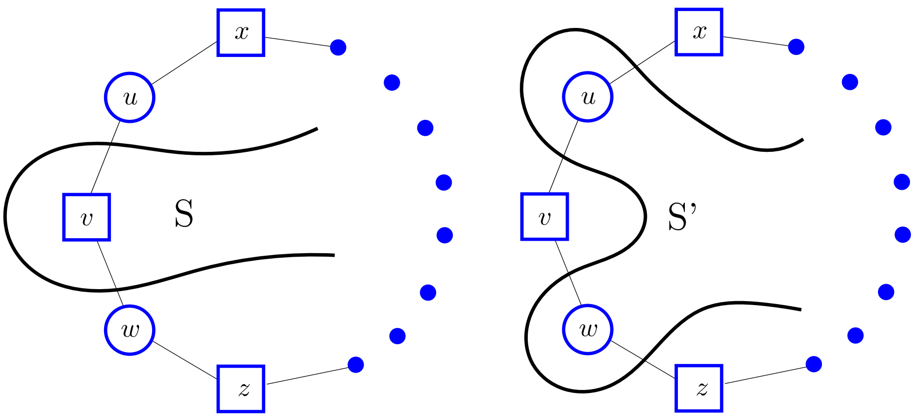


Figure 11: Lemma 5.11, Case 3

Because S is a stable set, we have $u, w \notin S$ in this case. We let $S' := (S \setminus \{v\}) \cup \{u, w\}$, as illustrated in Figure 11. Note that S' must be a separator in this case, because x and z must belong to different components of $G - S'$ as S is a separator.

Note that $G - S'$ contains one terminal component more than $G - S$, namely the component $\{v\}$, and that the terminal-free components of $G - S$ and of $G - S'$ are exactly the same. Hence, the lifted Steiner partition inequality for S' and $(n_1, \ldots, n_{n_S})$

$$\sum_{i \in S'} y_i - \sum_{i=1}^{n_S} y_{n_i} \geq t'_S - 1 = t_S$$

is contained in (PART) and satisfied by y. Adding (13) and plugging in (14) we obtain first

$$\sum_{i\in S'} y_i + y_v - y_u - y_w - \sum_{i=1}^{n_S} y_{n_i} \geq t_S - 1 + \varepsilon \quad \text{and then}$$

$$\sum_{i\in S} \bar{y}_i - \sum_{i=1}^{n_S} \bar{y}_{n_i} \geq t_S - 1 \ .$$

Hence, $\bar{y}$ fulfills the lifted Steiner partition inequality for S and $(n_1, \dots, n_{n_S})$ also in this case, which concludes the proof. □

Together, Lemma 5.10 and Lemma 5.11 imply that the equalities (8) hold for all irreducible optimal solutions of (PART) for all nonnegative objective functions. This now allows us to prove the main result of this section.

Theorem 5.12. *Let $G = (V, E)$ be a cycle with $V = \{v_1, \dots, v_{2k}, v_{2k+1} = v_0\}$, $E = \{v_i v_{i+1} \mid i = 1, \dots, 2k\}$, and $T = \{v_0, v_2, \dots, v_{2k-2}\}$ for $k \geq 2$. Then the lifted Steiner partition inequalities* (7) *and the non-negativity constraints $y_v \geq 0$ for all $v \in V$ completely describe the dominant of P. In other words, for each nonnegative objective function $c \in \mathbb{R}^V_{\geq 0}$ there exists an optimal solution of (PART) that is integer.*

Proof. For each nonnegative objective function c, there exists an optimal solution of (PART) that is irreducible. As we have seen in the previous two lemmata, any irreducible optimal solution satisfies $y_{v_i} = y_{v_{i-1}} + y_{v_{i+1}} - 1$ for all terminal nodes $v_i \in T$. Hence, it suffices to show that the linear program (PARTE) obtained by adding these equalities to (PART) has integer optimal solutions. This system reads

(PARTE)
$$\min c^T y$$
$$\sum_{v\in S} y_v - \sum_{i=1}^{n_S} y_{n_i} \geq t_S - 1 \quad \forall\, S \in \mathcal{S},\ (n_1, \dots, n_{n_S}) \in \mathcal{N}(S) \tag{15}$$
$$y_t - \sum_{v\in\Gamma^*(t)} y_v = -1 \qquad \forall\, t \in T \tag{16}$$
$$1 \geq y_v \geq 0 \qquad \forall v \in V \tag{17}$$

To prove that this formulation has integer optimal solutions, we show how the associated polyhedron is obtained from the minimum spanning tree polytope of a smaller graph. To this end, we consider the graph $G' = (V', E')$ with

$$V' := \{v_0, v_2, \dots, v_{2k-2}\} = T \quad \text{and}$$
$$E' := \{e_1, e_3, \dots, e_{2k-1}\} \quad \text{with} \quad e_i = v_{i-1}v_{i+1} \text{ for } i = 1, \dots, 2k-1.$$

This graph G' is a cycle on the k terminals of G, whose edges correspond to the non-terminal nodes. This allows us to associate each edge e_i with the corresponding non-terminal v_i. On this graph G', we define the edge-based objective function $c' \in \mathbb{R}^{E'}$ as

$$c'_{e_i} := c_{v_{i-1}} + c_{v_i} + c_{v_{i+1}} \quad \text{for all } e_i \in E'.$$

Obviously, $c' \geq 0$.

It is not hard to see that there is a simple correspondence between irreducible solutions for the NWDST problem on G and minimum spanning trees in G': Any given spanning tree $B' \subseteq E'$ in G' defines a connected dominating Steiner tree $I = I(B') := \{v_i \mid e_i \in B'\} \cup \{v_i \mid \text{both } e_{i-1}, e_{i+1} \in B'\}$ with objective $c(I) = c'(B') - c(T)$. Reversely, the set $I \subset V$ of (internal) nodes of any irreducible connected dominating Steiner tree in G defines a spanning tree $B' = B'(I) := \{e_i \mid v_i \in I \setminus T\}$ with objective $c'(B') = c(I) + c(T)$. Note that the objective values $c(I)$ and $c'(B')$ of a connected dominating Steiner tree and its corresponding spanning tree differ by $c(T)$, which is constant for any given problem instance. This allows us to derive the polyhedral description of the irreducible connected dominating Steiner trees in G from the polyhedral description of the minimum spanning trees in G'.

Given a partition $V_1 \dot\cup \ldots \dot\cup V_k = V$ of the nodes of some graph $G = (V, E)$, we denote by $\delta(V_1, \ldots, V_k) := \{uv \in E \mid u \in V_i,\, v \in V_j \text{ for } i \neq j\}$ the set of edges between different node sets of the partition. It has been shown in [4] that the minimum spanning tree polytope for any graph G is completely described by the boundary constraints $1 \geq x_e \geq 0$ for all $e \in E$ and the Steiner partition inequalities $\sum_{e \in \delta(V_1, \ldots, V_k)} x_e \geq k - 1$ for all partitions $V_1 \dot\cup \ldots \dot\cup V_k = V$. In fact, it is sufficient to consider so-called valid partitions, where each component V_i is connected. Partition inequalities corresponding to non-valid partitions are redundant. Applying this result to our graph G', we find that the following linear program is integral, i.e., has an integer optimal solution for each objective function:

$$\begin{aligned} \min \sum_{e \in E'} c'_e y_e & \\ \sum_{e \in \delta(V'_1, \ldots, V'_k)} y_e \geq k - 1 & \qquad \forall \text{ valid partitions } V'_1 \dot\cup \ldots \dot\cup V'_k = V' \\ 1 \geq y_e \geq 0 & \qquad \forall\, e \in E' \end{aligned}$$

Interpreting the edges e_i of G' as non-terminal nodes v_i in G, valid partitions $V'_1 \dot\cup \ldots \dot\cup V'_k$ of the nodes of G' can be interpreted as separator node sets $S \subseteq V \setminus T$ in G. Given a valid partition $V'_1 \dot\cup \ldots \dot\cup V'_k = V'$, the corresponding separator is $S := \{v_i \mid e_i \in \delta(V'_1, \ldots, V'_k)\}$ and the number of resulting components is k. Note that each of the resulting components is a terminal component and that each (terminal) node set V'_i is fully contained in one of these components. Reversely, any subset $S \subseteq V \setminus T$ defines a partition $V'_1 \dot\cup \ldots \dot\cup V'_k$ of V' with $k = t_S$. Hence, we can equivalently write the above linear program as follows:

$$\begin{aligned} \min \sum_{v \in V \setminus T} c'_v y_v & \\ \sum_{v \in S} y_v \geq t_S - 1 & \qquad \forall\, S \subseteq V \setminus T \\ 1 \geq y_v \geq 0 & \qquad \forall\, v \in V \setminus T \end{aligned}$$

Clearly, also this linear program is integer. However, it only contains the variables for the non-terminal nodes $v \in V \setminus T$. Introducing the missing variables y_t for the terminal nodes $t \in T$ together with the equalities $y_t - \sum_{v \in \Gamma^*(t)} y_v = -1$

linking them to the variables of their neighboring non-terminal nodes and adding the constant $-c(T)$ to the objective, we obtain the following linear program:

$$\text{(MT)} \qquad \min \sum_{v \in V \setminus T} c'_v y_v - c(T) \quad \big(= \sum_{v \in V} c_v y_v \big) \tag{18}$$

$$\sum_{v \in S} y_v \geq t_S - 1 \qquad \forall\, S \subseteq V \setminus T \tag{19}$$

$$x_t - \sum_{v \in \Gamma^*(t)} x_v = -1 \qquad \forall\, t \in T \tag{20}$$

$$1 \geq y_v \geq 0 \qquad \forall\, v \in V$$

Note that the newly introduced variables y_t for $t \in T$ neither occur in the objective function (with respect to c') nor in the partition inequalities (19). In fact, the variable y_t for terminal node $t \in T$ only occurs in its boundary constraints and in the equality (20) linking it to its two neighboring non-terminals. Given integer values for the variables y_v for $v \in V \setminus T$, one can thus set the values of the y_t for $t \in T$ in such a way that (20) and the boundary constraints are satisfied. Hence, also the linear program (MT) has an integer optimal solution for each objective function c'.

For nonnegative objective functions c, also c' is nonnegative. Hence, as shown in the previous lemmas, the irreducible solutions y of (PARTE), which coincide with the irreducible solutions of (MT), satisfy the additional equalities (8), i.e., we have $y_v = y_{v-1} + y_{v+1} - 1$ for all $v \in T$. This immediately implies that these solutions y also satisfy $\sum_{v \in V \setminus T} c'_v y_v = \sum_{v \in V} c_v y_v + c(T)$. Hence, (MT) has integer optimal solutions also for the objective $c^T y$ if c is nonnegative.

To conclude the proof, we now show that (MT) and (PARTE) are equivalent and define the same set of feasible solutions. Obviously, both models contain the same boundary constraints and the same equalities (16) and (20), respectively. Furthermore, the inequalities of type (19) are a subset of the larger class of lifted Steiner partition inequalities (15), which implies that (MT) is a relaxation of (PARTE). Hence, it suffices to show that all lifted Steiner partition inequalities (15) are implied by the constraints of (MT).

So, let $S \in \mathcal{S}$ and $(n_1, \ldots, n_{n_S}) \in \mathcal{N}(S)$ and consider the lifted Steiner partition inequality (15) defined by S and $(n_1, \ldots, n_{n_S})$. Due to Lemma 5.8 we may assume without loss of generality that S is a stable set, as otherwise the inequality would be redundant.

Let $S^+ := \{v \in \Gamma^*(t) \mid t \in S \cap T\}$ be the set of nodes that are adjacent to a terminal node in S. Since S is stable, we have $S^+ \cap S = \emptyset$. Because G is a cycle, nodes in S^+ may be neighboring either one or two terminal nodes in S. For simplicity, we denote by $S_2^+ := \{v_i \in S^+ \mid v_{i-1}, v_{i+1} \in S\}$ the set of nodes that are adjacent to two terminal nodes is S.

Finally, let $S' := S \setminus (S \cap T) \cup S^+$ be the separator obtained by replacing all terminals in S by their two neighbors. Obviously, we have $t'_S = t_S + |S \cap T|$. Also, as terminals and non-terminals alternate on the cycle G, the terminal-free components of $G - S$ must be exactly the single nodes in S_2^+, i.e., $(n_1, \ldots, n_{n_S})$ is just some ordering of the nodes in S_2^+. Since $S' \subseteq V \setminus T$, the corresponding partition inequality (19)

$$\sum_{v \in S'} y_v \geq t'_S - 1$$

is part of (MT). Adding the equalities (20) for all $t \in S \cap T$, one obtains

$$\sum_{v \in S'} y_v - \sum_{v_i \in S \cap T} (y_i - y_{i-1} - y_{i+1}) \geq t'_S - 1 - |S \cap T| = t_S - 1 \ .$$

Rearranging terms, we get

$$\Big(\sum_{v \in S'} y_v + \sum_{v \in S^+} y_v - \sum_{v \in S \cap T} y_v\Big) - \sum_{v \in S_2^+} y_v \geq t_S - 1 \ ,$$

which finally leads to

$$\sum_{v \in S} y_v - \sum_{i=1}^{n_S} y_{n_i} \geq t_S - 1 \ .$$

Hence, the lifted Steiner partition inequality for S and $(n_1, \ldots, n_{n_S})$ is implied by the constraints of (MT).

Consequently, both (MT) and (PARTE) describe the same set of solutions and, as (MT) has integer optimal solutions, so does (PARTE). □

We can extend the characterization to instances in circles where more than one non-terminals form a path. We say that a non-terminal v *lies between* the terminals t_1 and t_2 if walking along the circle in both possible directions, these are the first terminals we encounter.

Lemma 5.13. *Let $G = (V, E)$ be a cycle, $T \subset V$ be a terminal set that forms a stable set in G (i.e., no two terminals are adjacent), and $c : \mathbb{R}^V_{\geq 0}$ be nonnegative. Then, for any component C of $G - T$ and any irreducible integer solution y, all nodes $v \in C$ have the same value y_v.*

Proof. Note that the components of $G - T$ consist of non-terminal nodes only. Furthermore, any terminal node $t \in T$ is adjacent to exactly two components of $G - T$. Clearly, an integer solution y must choose all nodes of all but one component of $G - T$ in order to be feasible. Otherwise at least two nodes of two different components of $G - T$ would not be chosen, and these non-chosen nodes would define a terminal separator. With this observation, one easily verifies that an integer solution y is irreducible if and only if it chooses all nodes of G except for those of one component C of $G - T$ and the two terminal nodes adjacent to this component C. Hence, the variables of all non-terminals in the same component are either all equal to 1 or all equal to 0. □

In cycles where terminals are adjacent, the lifted partition inequalities are insufficient to fully describe the dominant of P. A simple example is given in Figure 5.1. In this example, the inequality $\sum_{x \in V} x_i \geq 3$ is needed in the description of the dominant of P, but it is not implied by the (lifted) Steiner partition inequalities.

However, for instances where terminals are adjacent, the solutions do not change structurally if we insert non-terminals with zero cost between every adjacent terminal-pair. One easily verifies that the optimal irreducible solutions for the modified instance including these extra non-terminals exactly correspond to

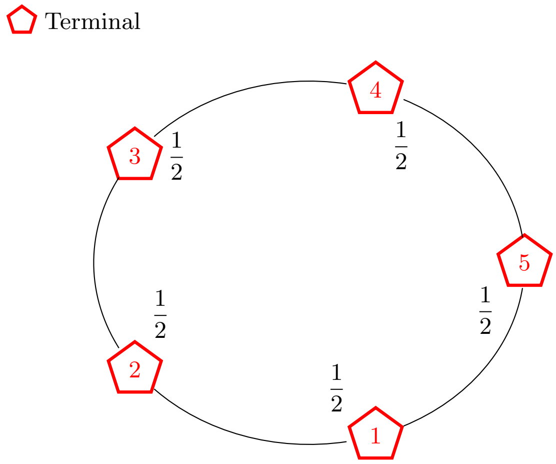


Figure 12: A valid basic solution for P that is not cut away by the lifted Steiner partition inequalities

optimal irreducible solutions of the original problem: An artificial non-terminal will be chosen if and only if one of its adjacent terminals is chosen.

Hence, we immediately get a complete description for the case with adjacent terminals using an extended formulation with at most twice as many variables.

Theorem 5.14. *The dominant of P is completely described by the lifted partition inequalities and the variable bounds in the case where the underlying graph G is a cycle and no two terminals are adjacent.*

In the case where adjacent terminals exist, these inequalities yield a complete description of an extended formulation, where we lift in one artificial non-terminal variables for each adjacent terminal pair.

In the case where adjacent terminals exist, one can construct a complete description in the original variable space by projecting this extended description back onto the original variable space in a straightforward way, using Fourier-Motzkin-Elimination for example.

5.2 Indegree inequalities

In this section, we will derive a connection between the lifted Steiner partition inequalities and the indegree inequalities. The well-known indegree inequalities are valid for the description of connected subgraphs. Let $V = \{1, \ldots, n\}$ and $d \in \mathbb{R}^n$. We call d an *indegree-vector* if there is an orientation O of G such that for this orientation, d_i is the vector of indegrees of G oriented by O. For each such vector d, the corresponding indegree inequality

$$\sum_{i \in V} (1 - d_i) y_i \leq 1$$

is then valid. Korte, Lovász, and Schrader [19] have shown that these inequalities induce all nontrivial facets of the connected subgraph polytope when G is a tree. Further conditions under which these inequalities are facet defining for the connected subgraph polytope have been studied by Wang, Buchanan, and Butenko [21].

For the NWDSTP polytope P on a cycle graph, we will now show that the indegree inequalities are implied by the other inequalities studied in this paper.

Theorem 5.15. *If $G = (V, E)$ is a simple cycle, the indegree inequalities are implied by the lifted Steiner partition inequalities* (7)*.*

Proof. Let $G = (V, E)$ denote a simple cycle and let O be an orientation of G. Let d denote the indegree vector corresponding to O. Let $V := V_0 \dot\cup V_1 \dot\cup V_2$, where $V_i := \{v \in V \mid d_v = i\}$. We set $S := V_2$, and, as above, let t_S be the number of terminal components in $G - S$. Because G is a simple cycle, every component of $G - S$ contains exactly one node from V_0. We set these nodes as the n_i. Let $T(S)$ be the terminal components from $G - S$. The resulting lifted partition inequality is given as:

$$\sum_{i \in V_2} y_i \geq \sum_{i \in V_0 \setminus T(S)} y_{n_i} + t_S - 1.$$

After adding $\sum_{i \in V_1 \cup V_2} y_i$ to both sides, we obtain:

$$\sum_{i \in V_2} 2y_i + \sum_{i \in V_1} y_i \geq \sum_{i \in V_1 \cup V_2} y_i + \sum_{i \in V_0 \setminus T(S)} y_{n_i} + t_S - 1.$$

Finally, after rewriting, we have:

$$\sum_{i \in V} d_i y_i = \sum_{i \in V_2} 2y_i + \sum_{i \in V_1} y_i + \sum_{i \in V_0} 0 \cdot y_i \geq \sum_{i \in V_1 \cup V_2 \cup (V_0 \setminus T(S))} y_i + t_S - 1 \geq \sum_{i \in V} y_i - 1$$

which concludes the proof. □

6 Conclusion

In this article we introduced the Node Weighted Dominating Steiner Tree Problem, a generalization of the Minimum Connected Dominating Set Problem in graphs, in which a least cost subset of nodes is to be found such that the given set of terminals is dominated. We provided a “thin” integer programming model that uses node variables only, and requires polynomial separation of an exponential number of connectivity cuts. For the underlying polytope, we gave the conditions under which these inequalities are facet defining. In the second half of the paper, we introduced a new family of Steiner partition inequalities and showed how to lift them to provide stronger linear programming relaxation bounds. For the special case when the input graph is a cycle, we showed that the dominant of the underlying polytope is either integral, or can be lifted and projected to easily obtain integral solutions.

Finally, we believe that for this new and challenging problem which is of particular importance in the design of communication networks dealing with cloud services, much more studies need to be done. On the one hand, exact methods for handling the large-scale instances are needed. The practical relevance and computational strength of the minimal node-separator inequalities studied in this paper has been demonstrated in [8], where these inequalities are shown to play an important role for the related Steiner tree problems. However, for the new families of (lifted) Steiner partition inequalities introduced in this paper, it remains an open question how they can influence the performance of branch-and-cut algorithms. On the other hand, further polyhedral studies could support the development of exact methods, by investigating other families of valid inequalities that strengthen the LP-relaxation bounds.

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