On the Asymmetric Connected Facility Location Polytope

Markus Leitner[1], Ivana Ljubić[1], Juan-José Salazar-González[2], and Markus Sinnl[1]

[1] ISOR, University of Vienna, Vienna, Austria
{markus.leitner,ivana.ljubic,markus.sinnl}@univie.ac.at
[2] DEIOC, Universidad de La Laguna, La Laguna, Spain
jjsalaza@ull.es

Abstract. This paper is concerned with the connected facility location problem, which has been intensively studied in the literature. The underlying polytopes, however, have not been investigated. This work is devoted to the polytope associated with the asymmetric version of the problem. We first lift known facets of the related Steiner arborescence and of the facility location polytope. Then we describe other new families of facet-inducing inequalities. Finally, computational results are reported.

1 Introduction

In the last years, the *connected facility location (ConFL) problem* and variants of it have received considerable attention from the operations research community (see, e.g., [1,2] and the references therein). The problem is of practical importance, e.g., in telecommunications, to model the deployment of *fiber-to-the-curb* networks, or in the design of data management networks. In this paper we are dealing with an *asymmetric ConFL* (aConFL): Given an assignment graph with connections between a set of *customers* and a set of potential *facility locations*, and a directed graph connecting facilities with each other using a (potentially empty) set of *intermediate nodes*, the goal of aConFL is to decide which facilities to open, how to assign customers to facilities and how to connect all open facilities to a *dedicated root node* at minimum cost.

Despite the large body of work on the ConFL, to our knowledge, there are no results on the facial structure of the underlying polytopes of ConFL. Our work is a first polyhedral study on aConFL. Our motivation for studying aConFL is twofold: 1) in some practical applications traversal of an edge in two opposite directions may involve different costs, and 2) the best performing computational approaches to ConFL are based on directed reformulations. In this paper we prove that some of these inequalities used in previous computational studies are facet-defining, and derive some new families of facet-defining inequalities. The obtained theoretical results are supported by a computational study on a newly generated benchmark set of digraphs. Polyhedral results for the symmetric version of the problem and a more extensive computational study are in [3].

The remainder of this article is organized as follows. Section 2 contains our main results considering the facet-defining inequalities of the aConFL polytope. In Section 3 we present our computational study.

More formally, in aConFL we are given a directed graph $D = (V, A_S, A_J)$ where the node set $V = S \cup J \cup \{0\}$ is the disjoint union of *Steiner nodes* S, customer nodes J and a dedicated root node 0. Facility nodes $I \subseteq S$ can be used to open facilities in which case facility opening costs $f_i \geq 0$, $\forall i \in I$, incur. For later use, let $K := S \setminus I$ denote the intermediate nodes that cannot be used as facilities, and let $K_0 := K \cup \{0\}$ and $S_0 = S \cup \{0\}$. The arc set $A_S \subseteq \{(s, t) : s, t \in V \setminus J\}$ represents possible connections between Steiner nodes. The arc set $A_J \subseteq \{(i, j) : i \in I, j \in J\}$ represents possible assignments of customers to facilities. In the context of telecommunication, A_S represents potential fiber optic connections in the core network, and A_J represents the copper cables connecting the customers to the core network through facilities. Arcs $a \in A_S$ are associated with establishing costs $c_a \geq 0$, $\forall a \in A_S$, and arcs $(i, j) \in A_J$ are associated with assignment costs $c_{ij} \geq 0$. The aConFL problem consists of selecting a subset of I of *open facilities*, connecting them through an *arborescence* rooted at 0 (that may use other Steiner nodes) and assigning each customer to exactly one open facility at the minimum cost.

In the following, we assume that $|I| \geq 3$ and $|J| \geq 3$. We also assume that subgraph $(S \cup \{0\}, A_S)$ (also called *core graph*) is a complete digraph in which all in-going arcs to the root node are removed, and that subgraph $(I \cup J, A_J)$ (also called *assignment graph*) is complete bipartite, i.e., $A_J = \{(i, j) : i \in I, j \in J\}$. Note that any instance of the undirected ConFL can be transformed into a aConFL instance by replacing each undirected core edge by a pair of oppositely directed arcs. Additionally, if no root node is given, both aConFL and ConFL can be transformed into a rooted aConFL instance by adding an artificial root node 0 together with arcs $(0, i)$, $\forall i \in I$, and additionally ensuring that the out-degree of this artificial root node is one.

2 The aConFL Polytope

We model aConFL using node decision variables $y_s \in \{0, 1\}$, $\forall s \in S$, which indicate if node s is part of the solution and facility decision variables $z_i \in \{0, 1\}$, $\forall i \in I$, which indicate whether facility i is opened. Furthermore, arc decision variables $x_a \in \{0, 1\}$, $\forall a \in A_S$, specify which arcs of the core graph are part of the directed arborescence, and assignment variables $a_{ij} \in \{0, 1\}$, $\forall i \in I$, $\forall j \in J$, indicate whether facility i serves customer j. Let $A = A_S \cup A_J$ denote the union of core and assignment arcs. For a set $H \subset V$, we define $\delta^-(H) := \{(u, v) \in A : u \notin H, v \in H\}$ and $\delta^+(H) := \{(u, v) \in A : u \in H, v \notin H\}$ and for sets $H, L \subset V$, we define $(H : L) := \{(u, v) \in A : u \in H, v \in L\}$. Moreover, for any vector $\mu \in \{0, 1\}^M$ over a ground set M, we write $\mu(M') = \sum_{m \in M'} \mu_m$, for any $M' \subseteq M$. The aConFL problem can now be formulated as follows:

$$\min \sum_{a\in A_S} c_a x_a + \sum_{i\in I} f_i z_i + \sum_{(i,j)\in A_J} c_{ij} a_{ij} \tag{1}$$

$$a(\delta^-(j)) = 1 \qquad \forall j \in J \tag{2}$$

$$a_{ij} \le z_i \qquad \forall i \in I, \forall j \in J \tag{3}$$

$$z_i \le y_i \qquad \forall i \in I \tag{4}$$

$$x(\delta^-(s)) = y_s \qquad \forall s \in S \tag{5}$$

$$x(\delta^-(H)) \ge y_s \qquad \forall H \subseteq S, \forall s \in H \tag{yCuts}$$

$$(x,y,z,a) \in \{0,1\}^{|A_S|+|S|+|I|+|A_J|} \tag{6}$$

Constraints (2) ensure that every customer is assigned to exactly one facility, while constraints (3) make sure that assignment arcs can only be used if the corresponding facility is opened. Inequalities (4) are the coupling constraints between node and facility variables and equations (5) link node variables to the set of arc variables corresponding to ingoing arcs. Together with the directed cutset constraints (yCuts) which ensure that there is a directed path from the root node 0 to every other node in the solution, they also ensure that the solution cannot contain cycles. Thus, the solution is a directed arborescence rooted at 0. Let $\mathcal{Q}$ denote the aConFL polytope, i.e.:

$$\mathcal{Q} = \text{conv}\{(x,y,z,a) \in \{0,1\}^{|A_S|+|S|+|I|+|A_J|} \mid (x,y,z,a) \text{ satisfies (2)–(6)}\}.$$

2.1 Dimension of the aConFL polytope

To establish the dimension of $\mathcal{Q}$ and some of its facet-inducing inequalities, we consider the intermediate polytopes

$$\mathcal{Q}_y(S') = \{(x,y,z,a) \in \mathcal{Q} : y_s = 1, s \in S'\} \qquad \forall S' \subseteq S.$$

The projection of $\mathcal{Q}_y(S')$ into the x space is the *Steiner arborescence polytope* $\mathcal{S}(S')$ with terminal set S' which has been studied by [4] The projection of $\mathcal{Q}_y(S')$ into the (z,a) space is the *facility location polytope* $\mathcal{U}$, with facilities I and customers J, studied in (for example) [5]. Since

$$\mathcal{Q}_y(S) = \mathcal{S}(S) \times \{y_s = 1, s \in S\} \times \mathcal{U},$$

the facets of $\mathcal{S}(S)$ and the facets of $\mathcal{U}$ are also the facets of $\mathcal{Q}_y(S)$. Since the dimension of $\mathcal{S}(S)$ is $|A_S| - |S|$ and the dimension of $\mathcal{U}$ is $|A_J| + |I| - |J|$, we have the following result:

Theorem 1. *$dim(\mathcal{Q}_y(S)) = |A_S| + |A_J| + |I| - |S| - |J|$.*

Besides, we can show that:

Theorem 2. *For all $S' \subseteq S$, $dim(\mathcal{Q}_y(S')) = |A_S| + |A_J| + |I| - |S'| - |J|$.*

The dimension of $\mathcal{Q}$ immediately follows from Theorem 2:

Corollary 1. $dim(\mathcal{Q}) = |A_S| + |A_J| + |I| - |J|$.

The proof of Theorem 2 also reveals that the following family of valid inequalities is facet-inducing.

Theorem 3. *Inequalities $y_s \leq 1$ are facet-inducing for every $s \in S$.*

Proof. The face induced by $y_s = 1$ corresponds to $S' = \{s\}$ in the proof of Theorem 2. □

2.2 Facets obtained by Lifting

The proof of Theorem 2 shows that removing a node from S', $S' \subseteq S$, increases the dimension of $\mathcal{Q}_y(S')$ by one. Thus, the facet-defining inequalities given in this section can be obtained by lifting (see, e.g., [6]).

Theorem 4 provides facet-inducing inequalities obtained by lifting from the facility location polytope $\mathcal{U}$ and using the results of [7]. Let $\mathcal{H}$ be a family of injective mappings $h : I \mapsto J$, for $|I| \leq |J|$, i.e., $i \neq i' \Rightarrow h(i) \neq h(i')$.

Theorem 4.

(a) Inequalities $a_{ij} \leq z_i$ are facet-inducing for all $i \in I$ and all $j \in J$.
(b) Inequalities $a_{ij} \geq 0$ are facet-inducing for all $i \in I$ and all $j \in J$.
(c) Inequalities $z_i \leq y_i$ are facet-inducing for all $i \in I$.
(d) Let $|I| \leq |J|$ and let $h \in \mathcal{H}$ be an injective mapping. Then inequalities $\sum_{i \in I} a_{ih(i)} + z_i \geq 2$ are facet-inducing.

The following facet-defining inequalities can be obtained by lifting from the Steiner arborescence polytope $\mathcal{S}(S)$.

Theorem 5.

(a) Inequalities $x_{st} \geq 0$ are facet-inducing for all $(s,t) \in A_S$, if $s \neq 0$ or $|S| \geq 4$.
(b) Inequalities $x(\delta^-(H)) \geq y_s$ are facet-inducing for all $H \subseteq S$ with $|H| \geq 2$, $|H \cap I| \leq |I| - 1$ and $s \in H$.
(c) Inequalities $x(\delta^-(H)) \geq 1$ are facet-inducing for all $H \subseteq S$, $|H \cap I| = |I|$.

2.3 New Facets

For proving that some additional inequalities are facet-inducing, the following well-known result (restated appropriately for our formulation) will be used:

Lemma 1. [6] *Let $(A^=, b^=)$ be the equality set of $\mathcal{Q}$, denote its size with m, and let $\mathcal{F} = \{(x,y,z,a) \in \mathcal{Q} : \pi_x x + \pi_y y + \pi_z z + \pi_a a = \pi_0\}$ be a proper face of $\mathcal{Q}$. Then the following two statements are equivalent:*

1. *$\mathcal{F}$ is a facet of $\mathcal{Q}$.*

2. *If* $\mathcal{F} \subseteq \mathcal{G} = \{(x, y, z, a) \in \mathcal{Q} : \{\alpha x + \beta y + \gamma z + \delta a = \lambda_0\}$*, then* $(\alpha, \beta, \gamma, \delta, \lambda_0) = (s(\pi_x, \pi_y, \pi_z, \pi_a) + tA^{=}, s\pi_0 + tb^{=})$*, for some* $s \in \mathbb{R}$ *and* $t \in \mathbb{R}^m$*.*

In the following proofs we will construct feasible solutions $L \in \mathcal{F}$ for the face $\mathcal{F}$ under consideration and insert them into the equality defining $\mathcal{G}$ in order to determine the coefficients of $\mathcal{G}$ and then use Lemma 1. We denote the left-hand side of $\mathcal{G}$, i.e., the evaluation of $\alpha x + \beta y + \gamma z + \delta a$ for some L by $\mathcal{L}(L)$.

First, we consider the inequalities

$$x(\delta^-(H)) + \sum_{i \notin I \cap H} a_{ij} \geq 1, \quad \forall j \in J, \forall H \subseteq S \tag{aCuts}$$

These inequalities, already considered in [1] for ConFL, state that for each customer j and any subset H of the core nodes (excluding 0), j is either served by a facility outside of H, or there has to be an arc going into H. Note that when $|H \cap I| = |I|$, the inequalities reduce to $x(\delta^-(H)) \geq 1$.

Theorem 6. *Inequalities* (aCuts) *are facet-inducing iff* $2 \leq |H \cap I| \leq |I| - 1$*.*

Proof. Let $\mathcal{F} = \{(x, y, z, a) \in \mathcal{Q} : x(\delta^-(H)) + \sum_{i \notin I \cap H} a_{ij} = 1\}$ be the proper face induced by (aCuts) for some $j \in J$ and $H \subset S$, $2 \leq |H \cap I| \leq |I| - 1$. In the following, we describe feasible solutions as tuples $L_q = (V_q \cap H, V_q \setminus H, I_q, A_q \cap A_S, A_q \cap A_J)$. Thereby, $V_q \subseteq S$ is the set of core nodes of solution L_q, $I_q \subseteq I$ its set of open facilities and $A_q \subset A_S \cup A_J$ its set of arcs. For the rest of the proof, we will need the following feasible solutions from $\mathcal{F}$, where $i, i', i_1, i_2 \in I$, $s, s_1, s_2, t, t_1, t_2 \in S$, and $j' \in J$, $j' \neq j$. Note that in some solutions, we make use of the assumption $|H \cap I| \geq 2$.

- $L_1 = (\{i_1, i_2\}, \emptyset, \{i_1\}, \{(0, i_1), (i_1, i_2)\}, (i_1 : J))$
- $L_2 = (\{i_1, i_2\}, \emptyset, \{i_1, i_2\}, \{(0, i_1), (i_1, i_2)\}, (i_1 : J))$
- $L_3 = (\{i, s\}, \emptyset, \{i\}, \{(0, i), (i, s)\}, (i : J))$
- $L_4 = (\{i, s\}, \{t\}, \{(0, i), (i, s), (s, t)\}, (i : J))$
- $L_5 = (\{i\}, \emptyset, \{i\}, \{(0, i)\}, (i : J))$
- $L_6 = (\{i_1, i_2\}, \emptyset, \{i_1, i_2\}, \{(0, i_1), (i_1, i_2)\}, \{(i_2, j'), (i_1 : J \setminus \{j'\}\}))$
- $L_7 = (\{i_1, i_2\}, \emptyset, \{i_1, i_2\}, \{(0, i_1), (i_1, i_2)\}, (i_1 : J))$
- $L_8 = (\{i, t\}, \{s_1\}, \{i\}, \{(0, s_1), (s_1, t), (t, i)\}, (i : J))$
- $L_9 = (\{i, t\}, \{s_2\}, \{i\}, \{(0, s_2), (s_2, t), (t, i)\}, (i : J))$
- $L_{10} = (\{i, t_1\}, \{s\}, \{i\}, \{(0, s), (s, t_1), (t_1, i)\}, (i : J))$
- $L_{11} = (\{i, t_2\}, \{s\}, \{i\}, \{(0, s), (s, t_2), (t_2, i)\}, (i : J))$
- $L_{12} = (\emptyset, \{i'\}, \{i'\}, \{(0, i')\}, (i' : J))$ (assumption: $|H \cap I| \leq |I| - 1$)

We now suppose $\mathcal{F} \subseteq \mathcal{G}$ and determine the coefficients of $\mathcal{G}$.

(a) $\gamma_i = 0$, $\forall i \in I$: If $i \in H$, this follows from $\mathcal{L}(L_1) = \mathcal{L}(L_2)$. Else, it is obtained from $\mathcal{L}(L_{1'}) = \mathcal{L}(L_{2'})$, where $L_{1'}$ and $L_{2'}$ are obtained from L_1 and L_2, respectively, by assuming $i_2 \in I \setminus H$.

(b) $\alpha_{st} = -\beta_t$, $\forall s \in H$, $\forall t \in S \setminus H$: Obtained from $\mathcal{L}(L_3) = \mathcal{L}(L_4)$.

(c) $\alpha_{st} = -\beta_t$, $\forall s, t \in H$: If $|H| \geq 3$, the relation is obtained from $\mathcal{L}(L_3) = \mathcal{L}(L_{4'})$ where $L_{4'}$ is obtained from L_4 by assuming $t \in H$. Else, $\{s, t\} \subseteq I$ by assumption and the relation is obtained from $\mathcal{L}(L_5) = \mathcal{L}(L_{5'})$ for $i = s$ and where $L_{5'}$ is obtained from L_5 by adding node $t \in H$ and arc (i, t).
(d) $\alpha_{st} = -\beta_t$, $\forall s \in S_0 \setminus H$, $\forall t \in S \setminus H$: If $s \neq 0$, this is follows from $\mathcal{L}(L_{3'}) = \mathcal{L}(L_{4'})$ where $L_{3'}$ and $L_{4'}$ are obtained from L_3 and L_4, respectively, by assuming $s \in S \setminus H$. For $s = 0$, compare an arbitrary solution with a variant of it additionally considering a new node $t \in S \setminus H$ and arc $(0, t)$.
(e) $\delta_{ij'} = \delta^H_{j'}$, $\forall i \in H \cap I$, $\forall j' \in J$: Obtained from $\mathcal{L}(L_6) = \mathcal{L}(L_7)$
(f) $\delta_{ij'} = \delta^H_{j'}$, $\forall i \in I \setminus H$, $\forall j' \in J$, $j' \neq j$: Obtained from $\mathcal{L}(L_{6'}) = \mathcal{L}(L_{7'})$, where $\mathcal{L}(L_{6'})$ and $\mathcal{L}(L_{7'})$ are obtained from $\mathcal{L}(L_6)$ and $\mathcal{L}(L_7)$, respectively, by assuming $i_2 \in I \setminus H$. Note that $j' \neq j$ must hold since, neither $\mathcal{L}(L_{6'})$ nor $\mathcal{L}(L_{7'})$ would lie on $\mathcal{F}$ otherwise.
(g) $\alpha_{s_1 t} = \alpha_{s_2 t}$, $\forall s_1, s_2 \in S_0 \setminus H$, $\forall t \in H$: Obtained from $\mathcal{L}(L_8) = \mathcal{L}(L_9)$ using the result from (d).
(h) $\alpha_{st_1} + \beta_{t_1} = \alpha_{st_2} + \beta_{t_2}$, $\forall s \in S_0 \setminus H$, $\forall t_1, t_2 \in H$: If $s \neq 0$, the result follows from $\mathcal{L}(L_{10}) = \mathcal{L}(L_{11})$ using the result from (c). For $s = 0$ consider variants of L_{10} and L_{11} obtained by contracting arc $(0, s)$.
(i) $\alpha_{st} + \beta_t = \rho$, $\forall s \in S_0 \setminus H$, $\forall t \in H$: Follows from (g) and (h).
(j) $\delta_{i'j} = \rho + \delta^H_j$, $\forall i' \in I \setminus H$: Obtained from $\mathcal{L}(L_5) = \mathcal{L}(L_{12})$ using the results of (e),(f), and (i).

Inserting the obtained coefficients in the equation defining $\mathcal{G}$, using equations (5) and inserting an arbitrary solution from $\mathcal{F}$ (yielding $\lambda_0 = \rho + \sum_{j \in J} \delta^I_j$), the equation can be simplified to

$$\rho(x(\delta^-(H)) + \sum_{i \notin I \cap H} a_{ij}) + \sum_{j \in J} \delta^H_j a(\delta^-(j)) = \rho + \sum_{j \in J} \delta^H_j$$

which is a linear combination of the equation defining $\mathcal{F}$ and equations (2).
To show that $|H \cap I| \geq 2$ is also a necessary condition, observe that when $|H \cap I| = 1$, inequalities (aCuts) are dominated by $x(\delta^-(H)) \geq y_i$. For $|H \cap I| = 0$, the inequality is the sum of trivial facets $x_a \geq 0$, for $a \in \delta^-(H)$ and the equation $a(\delta^-(j)) = 1$. $\square$

Theorem 7. *Let $h \in \mathcal{H}$ be an injective mapping and $\hat{I} \subset I$. Then, the following inequalities are valid for aConFL:*

$$z(\hat{I}) + \sum_{i \in I \setminus \hat{I}} (y_i + a_{ih(i)}) + x(K_0 : \hat{I}) \geq 2 \tag{7}$$

Proof. If $z(\hat{I}) \geq 2$ or $y(I \setminus \hat{I}) \geq 2$ the theorem holds. Since at least one facility needs to be opened, it is sufficient to consider the following two cases:
1) $z(\hat{I}) = 0$ and $z(I \setminus \hat{I}) = 1$, i.e., there exists a unique facility $i \in I \setminus \hat{I}$ with $z_i = 1$ to which all customers are assigned. Then, validity is implied by $a_{ih(i)} = 1$.
2) $z(\hat{I}) = 1$ and $y(I \setminus \hat{I}) = 0$. Let $i \in \hat{I}$ be the unique facility with $z_i = 1$. Since no nodes from $I \setminus \hat{I}$ are used ($y(I \setminus \hat{I}) = 0$), validity of the inequality follows since i must be connected to the root and thus $x(K_0 : \hat{I}) \geq 1$. $\square$

Theorem 8. *Inequalities* (7) *are facet-inducing iff* $\hat{I} \neq \emptyset$ *and* $|I \setminus \hat{I}| \geq 2$.

Proof. For some $\hat{I} \neq \emptyset$, such that $|I \setminus \hat{I}| \geq 2$, let $\mathcal{F} = \{(x, y, z, a) \in \mathcal{Q} : z(\hat{I}) + \sum_{i \in I \setminus \hat{I}}(y_i + a_{ih(i)}) + x(K_0 : \hat{I}) = 2\}$ be the proper face induced by (7).

In the following, we describe feasible solutions as tuples $L_q = (V_q \cap \hat{I}, V_q \cap (I \setminus \hat{I}), V_q \cap K, I_q, A_q \cap A_S, A_q \cap A_J)$. Thereby, $V_q \subseteq S \cup \{0\}$ is the set of core nodes of solution L_q, $I_q \subseteq I$ its set of open facilities and $A_q \subset A_S \cup A_J$ its set of arcs. For the proof, we will need the following feasible solutions from $\mathcal{F}$, where $i_1, i_2, i_3 \in I$, $s_1, s_2, t \in K$. For solutions in which exactly two facilities, say $i, i' \in I$, are open, let $\mathcal{A} = (i : J \setminus \{h(i)\}) \cup \{(i', h(i))\}$ be an assignment of customers, s.t. the sum of a-variables in (7) is zero.

- $L_1 = (\{i_1\}, \{i_2\}, \emptyset, \{i_1, i_2\}, \{(0, i_2), (i_2, i_1)\}, (i_1 : J))$
- $L_2 = (\{i_1\}, \{i_2\}, \emptyset, \{i_1\}, \{(0, i_2), (i_2, i_1)\}, (i_1 : J))$
- $L_3 = (\{i_1\}, \{i_2\}, \emptyset, \{i_1, i_2\}, \{(0, i_2), (i_2, i_1)\}, \mathcal{A})$
- $L_4 = (\{i_1\}, \emptyset, \emptyset, \{i_1\}, \{(0, i_1)\}, (i_1 : J))$
- $L_5 = (\{i_1\}, \{i_2\}, \emptyset, \{i_2\}, \{(0, i_2), (i_2, i_1)\}, (i_2 : J))$
- $L_6 = (\{i_1\}, \emptyset, \{s_1\}, \{i_1\}, \{(0, s_1), (s_1, i_1)\}, (i_1 : J))$
- $L_7 = (\{i_1\}, \emptyset, \{s_2\}, \{i_1\}, \{(0, s_2), (s_2, i_1)\}, (i_1 : J))$
- $L_8 = (\{i_2\}, \emptyset, \{s_1\}, \{i_2\}, \{(0, s_1), (s_1, i_2)\}, (i_2 : J))$
- $L_9 = (\emptyset, \{i_2\}, \emptyset, \{i_2\}, \{(0, i_2)\}, (i_2 : J))$
- $L_{10} = (\emptyset, \{i_1, i_2\}, \emptyset, \{i_1, i_2\}, \{(0, i_1), (0, i_2)\}, \mathcal{A})$
- $L_{11} = (\emptyset, \{i_1, i_2\}, \emptyset, \{i_1, i_2\}, \{(0, i_1), (i_1, i_2)\}, \mathcal{A})$
- $L_{12} = (\{i_1\}, \{i_2, i_3\}, \emptyset, \{i_2, i_3\}, \{(0, i_2), (i_2, i_3), (i_2, i_1)\}, \mathcal{A})$

We now suppose $\mathcal{F} \subseteq \mathcal{G}$ and determine the coefficients of $\mathcal{G}$.

(a) $\gamma_i = 0$, $\forall i \in I \setminus \hat{I}$: Obtained from $\mathcal{L}(L_1) = \mathcal{L}(L_2)$.
(b) $\delta_{ij} = \delta_j^I$, $\forall i \in \hat{I}$, $\forall j \in J$ and $\forall i \in I \setminus \hat{I}$, $j \in J$, $j \neq h(i)$: From $\mathcal{L}(L_1) = \mathcal{L}(L_3)$ it follows that all the coefficients δ_{ij} for a $j \in J$ (except when $j = h(i)$ for $i \in I \setminus \hat{I}$) are the same – the coefficient is denoted by δ_j^I.
(c) $\alpha_{st} = -\beta_t$, $\forall s \in S_0$, $\forall t \in K$: Obtained from $\mathcal{L}(L_s) = \mathcal{L}(L_{st})$ where L_s is an arbitrary solution lying on $\mathcal{F}$ containing $s \in S_0$, but not $t \in K$, and L_{st} is obtained from L_s by attaching arc (s, t) to it.
(d) $\alpha_{i'i} = -\beta_i$, $\forall i', i \in \hat{I}$: Follows from $\mathcal{L}(L_{i'}) = \mathcal{L}(L_{i'i})$, where $L_{i'}$ is an arbitrary solution lying on $\mathcal{F}$ containing $i' \in \hat{I}$, but not $i \in \hat{I}$, and $L_{i'i}$ is obtained from $L_{i'}$ by attaching arc (i', i) to it.
(e) $\delta_{ih(i)} = \gamma^I + \delta_{h(i)}^I$ for $i \in I \setminus \hat{I}$: Obtained from $\mathcal{L}(L_2) = \mathcal{L}(L_5)$, which gives $\delta_{i_2 h(i_2)} = \gamma_{i_1} + \delta_{h(i_2)}^I$, where we used results from step (b). This result implies that $\gamma_{i_1} = \gamma_{i'}$ for $i_1, i' \in \hat{I}$. Denote this value by γ^I.
(f) $\alpha_{si} = \alpha_i^I$, $\forall i \in \hat{I}$, $s \in K$: Obtained from $\mathcal{L}(L_6) = \mathcal{L}(L_7)$, which gives $\alpha_{s_1 i_1} = \alpha_{s_1 i_2}$, where we used results from step (c). Thus, all arcs from K to a facility $i \in \hat{I}$ have the same coefficient, denote it by α_i^I.
(g) $\alpha_{0i} = \alpha_i^I$, $\forall i \in \hat{I}$: Obtained from $\mathcal{L}(L_4) = \mathcal{L}(L_6)$.
(h) $\alpha_i^I + \beta_i = \rho$, $\forall i \in \hat{I}$: Obtained from $\mathcal{L}(L_6) = \mathcal{L}(L_8)$, which gives $\alpha_{i_1}^I + \beta_{i_1} = \alpha_{i_2}^I + \beta_{i_2}$, where we used results from steps (b), (c) and (f). Hence, this sum is a constant value for every node in $\hat{I}$ – denote it by ρ.

(i) $\alpha_{ti} + \beta_i = \rho$, $\forall i \in I \setminus \hat{I}$, $\forall t \in K_0 \cup \hat{I}$: We demonstrate this result for $t = 0$, and similar solutions can be constructed for $t \in K \cup \hat{I}$. From $\mathcal{L}(L_4) = \mathcal{L}(L_9)$ and using results of (a), (e), we have: $\alpha_{0i_1} + \beta_{i_1} + \gamma^I + \delta^I_{h(i_2)} = \alpha_{0i_2} + \beta_{i_2} + \delta_{i_2 h(i_2)}$. Using one more time the result of (e), we obtain: $\alpha_{0i_1} + \beta_{i_1} = \alpha_{0i_2} + \beta_{i_2}$ for all $i_1 \in \hat{I}$ and $i_2 \in I \setminus \hat{I}$. From (h), it follows that $\alpha_{0i} + \beta_i = \rho$, $\forall i \in I \setminus \hat{I}$.

(j) $\alpha_{i'i} + \beta_i = \rho$, $\forall i', i \in I \setminus \hat{I}$: From $\mathcal{L}(L_{10}) = \mathcal{L}(L_{11})$ it follows that $\alpha_{0i_2} = \alpha_{i_1 i_2}$. By adding β_{i_2} to both sides and using the result from (i), the result follows.

(k) $\rho = \gamma^I$: From $\mathcal{L}(L_{12}) = \mathcal{L}(L_3)$ and using results from (b), it follows that $\gamma_{i1} = \alpha i_2 i_3 + \beta_{i3}$. From (e) and (j), we have $\rho = \gamma^I$. Note that for this step we need that $|I \setminus \hat{I}| \geq 2$.

Inserting the obtained coefficients in the equation defining $\mathcal{G}$, we get

$$\rho z(\hat{I}) + \sum_{i \in \hat{I}} ((\rho - \beta_i) x(K_0 : i) - \beta_i x(I : i) + \beta_i y_i) + \sum_{k \in K} (-\beta_k x(\delta^-(k)) + \beta_k y_k) +$$
$$\sum_{i \in I \setminus \hat{I}} ((\rho - \beta_i) x(\delta^-(i)) + \beta_i y_i + \rho a_{ih(i)}) + \sum_{j \in J} \delta^I_j a(I : j) = \lambda_0$$

By inserting any of the used solutions into the left-hand-side of the equation, we get $\lambda_0 = 2\rho + \sum_{j \in J} \delta^I_j$. Using equations (5), the equation can be simplified to

$$\rho(z(\hat{I}) + x(K_0 : \hat{I}) + y(I \setminus \hat{I}) + \sum_{i \in I \setminus \hat{I}} a_{ih(i)}) + \sum_{j \in J} \delta^I_j a(\delta^-(j)) = 2\rho + \sum_{j \in J} \delta^I_j.$$

It can be seen that the equation is a linear combination of the equation defining $\mathcal{F}$ and equations (2). Thus, (7) are facet-inducing when $|I \setminus \hat{I}| \geq 2$, $\hat{I} \neq \emptyset$.

To see that $\hat{I} \neq \emptyset$ and $|I \setminus \hat{I}| \geq 2$ is also a necessary condition, consider the following cases:

1. For $\hat{I} = I$, the inequality reduces to $z(I) + x(K_0 : I) \geq 2$, which is dominated by $a(\delta^-(j) + x(K_0 : I) \geq 2$. The latter is a linear combination of a facet $x(\delta^-(I)) \geq 1$ and an equation of type (2).
2. For $\hat{I} = I \setminus \{i\}$, the inequality reduces to $z(I \setminus \{i\}) + a_{ih(i)} + y_i + x(K_0 : I \setminus \{i\}) \geq 2$. Notice that this inequality is dominated by the inequality in which y_i is replaced by $x(K_0 : i)$. The latter reduces to $z(I \setminus \{i\}) + a_{ih(i)} + x(K_0 : I) \geq 2$, which is also not facet-inducing, for the same reasons as above.
3. Finally, for $\hat{I} = \emptyset$, we obtain $\sum_{i \in I} (y_i + a_{ih(i)}) \geq 2$ which is dominated by facet-defining constraints $\sum_{i \in I} (z_i + a_{ih(i)}) \geq 2$. □

For the next family of valid inequalities, we employ a direct proof to show that the inequalities are facet-inducing.

Theorem 9. *Let $h \in \mathcal{H}$ be an injective mapping, and let $\hat{I} \subset I$, and $s \in K$. Then, the following inequalities are valid for aConFL:*

$$z(\hat{I}) + \sum_{i \in I \setminus \hat{I}} (y_i + a_{ih(i)}) + y_s + x(K_0 \setminus \{s\} : \hat{I} \cup \{s\}) \geq 2 + x(s : I \setminus \hat{I}) \qquad (8)$$

Proof. We will distinguish between the following cases:
1) $y_s = 0$: Inequality (8) corresponds to inequality (7) since $y_s = 0$ implies that $x(s : I \setminus \hat{I}) = 0$ and $x(K_0 \setminus \{s\} : \hat{I} \cup \{s\}) = x(K_0 : \hat{I})$.
2) $y_s = 1$ and $x(s : I \setminus \hat{I}) = 0$: Since at least one facility must be opened we obtain $z(\hat{I}) + \sum_{i \in I \setminus \hat{I}} (y_i + a_{ih(i)}) \geq 1$ which trivially holds.
3) $y_s = 1$ and $x(s : I \setminus \hat{I}) \geq 1$: Let $I' = \{i \in I \setminus \hat{I} \mid x_{si} = 1\}$ and observe that $\sum_{i \in I \setminus \hat{I}} y_i \geq \sum_{i \in I'} y_i \geq x(s : I \setminus \hat{I})$ due to (5). Further note that the path from 0 to s either contains at least one arc from the cut $(K_0 \setminus \{s\} : \hat{I} \cup \{s\})$ or at least one node $i' \in I \setminus \hat{I}$. In either case, validity of (8) follows immediately. □

Theorem 10. *Inequalities* (8) *are facet-inducing if* $|I \setminus \hat{I}| \geq 2$ *and* $\hat{I} \neq \emptyset$.

Proof. Let $\mathcal{F}$ be the face induced by (8) for given $\hat{I} \subset I$, $h \in \mathcal{H}$ and $s \in K$. We show how to construct $|A_S| + |A_J| + |I| - |J|$ affinely independent solutions lying on $\mathcal{F}$, which implies that $\mathcal{F}$ is a facet. We proceed in two steps, in the first step, we construct solutions that do not contain s and in the second step, we construct solutions containing s.
1) Let $D' = (V \setminus \{s\}, A'_S, A_J)$ be a digraph obtained by removing s from D. By Theorem 8, the corresponding inequality (7) (for the given h and $\hat{I}$) is facet-defining, and therefore we can determine $|A'_S| + |A_J| + |I| - |J|$ affinely independent solutions in the associated lower dimensional space. By setting $y_s = 0$ and $x_{0s} = x_{si} = x_{is} = 0$, for all $i \in S, i \neq s$, these solutions are extended to feasible and affinely independent solutions lying on $\mathcal{F}$. Therefore, it only remains to additionally construct $|A_S| - |A'_S| = 2|S| - 1$ affinely independent solutions lying on $\mathcal{F}$ such that $y_s = 1$. This is done in the next step.
2) The constructed solutions will be described using 6-tuples as in the proof of Theorem 8. Moreover, we will also use the assignment $\mathcal{A}$ defined in the same proof.

(a) Fix some facility $u \in \hat{I}$ and the arc $(u, s) \in A_S$. Now, pick some facility $i' \in I \setminus \hat{I}$ and for each $i \in I \setminus \hat{I}$, $i \neq i'$ build the following feasible solutions: $L_{si} = (\{u\}, \{i, i'\}, \{s\}, \{i, i'\}, \{(0, i'), (i', u), (u, s), (s, i)\}, \mathcal{A})$. Clearly, that way we create $|I \setminus \hat{I}| - 1$ affinely independent solutions due to the arcs (s, i). One more affinely independent solution can be found by switching the roles of i and i'.
(b) Consider now solutions $L_{is} = (\emptyset, \{i, i'\}, \{s\}, \{i, i'\}, \{(0, i), (i, s), (s, i')\}, \mathcal{A})$ for $i' \neq i \in I \setminus \hat{I}$. These solutions are all affinely independent due to the arcs (i, s). Again, we can also define a solution, where i and i' switch roles so that in total we obtain $|I \setminus \hat{I}|$ more affinely independent solutions.
(c) We create now solutions L_{sk}, for each $k \in K$, $k \neq s$, by adding arc (s, k) to one fixed solution L_{si} from step (a). That way, we obtain $|K| - 1$ affinely independent solutions. Moreover, for each $k \in K$, $k \neq s$, consider solutions $L_{ks} = (\emptyset, \{i, i'\}, \{s, k\}, \{i, i'\}, \{(0, k), (k, s), (s, i), (s, i')\}, \mathcal{A})$. We additionally obtain $|K| - 1$ affinely independent solutions due to arcs (k, s).
(d) One more solution is constructed as: $L_{0s} = (\emptyset, \{i, i'\}, \{s\}, \{i, i'\}, \{(0, s), (s, i), (s, i')\}, \mathcal{A})$. This solution is affinely independent from all the previous ones due to arc $(0, s)$.

(e) Take the solution from step (d) and construct solutions $L_{su'}$ for each $u' \in \hat{I}$ by adding arc (s, u') to L_{0s}. We obtain $|\hat{I}|$ affinely independent solutions this way.
(f) Consider now solutions $L_{u's} = (\{u'\}, \{i, i'\}, \{s\}, \{i, i'\}, \{(0, u'), (u', s), (s, i), (s, i')\}, \mathcal{A})$ for $u' \in \hat{I}$, $u' \neq u$, where u is the facility fixed for the solutions constructed in step (a). We get $|\hat{I}| - 1$ affinely independent solutions due to arcs (u', s).
(g) Note that all $2(|I \setminus \hat{I}| + |K| - 1 + |\hat{I}|) = 2|S| - 2$ solutions constructed so far are easily seen to be affinely independent, since in every solution, a previously unused arc is involved. The last affinely independent solution is constructed as $L^* = (\{u\}, I \setminus \hat{I}, \{s\}, \{i, i'\}, \{(0, u), (u, s), (s : I \setminus \hat{I})\}, \mathcal{A})$, where u is the facility from step (a). This concludes the proof. $\square$

3 Computational Results

3.1 Instances

To our knowledge, no instance sets with asymmetric costs are available for variants of ConFL or closely related problems. Thus, we generated two sets of random instances in the following way (following procedures described in [8,9]): $|V|$ points, each corresponding to one node in $S_0 \cup J$, are randomly generated in the Euclidean plane of size 100×100. Let (u_x, u_y) and (v_x, v_y) be the coordinates of two such nodes $u, v \in S_0 \cup J$ and let $\Delta_x(uv) = v_x - u_x$ and $\Delta_y(uv) = v_y - u_y$. Then, arc costs are defined as $c_{uv} = \omega \lfloor \sqrt{\Delta_x(uv)^2 + \xi \Delta_y(uv)^2} \rfloor$. Thereby, $\omega = 1$ for core arcs, $\omega = 3$ for assignment arcs, $\xi = 1$ if $\Delta_y \leq 0$ and $\xi = 2$ if $\Delta_y > 0$. Facility opening costs f_i, $\forall i \in I$ are integers chosen uniformly at random from the interval $[30, 60]$.

The first set of instances, denoted by A, consists of 20 complete graphs with $|I| = |J| = 100$ and $|K| = 50$. The second set of instances, denoted by B, consists of 20 sparse graphs with $|I| = |J| = 150$ and $|K| = 75$. In the latter instances, an arc between u and v only exists if the Euclidean distance between them is smaller than 40% of the largest Euclidean distance between any two points in this graph.

In addition to these randomly generated asymmetric instances, we also considered the symmetric Stein+UFL instances from [1]. The instances have $|I| = |J| = 200$ or $|I| = |J| = 300$, while $|S|$ ranges between 500 and 1000. Depending on the size of $|S|$, we get two sets of instances, denoted by C and D. In these instances the core network is sparse, while the assignment graph is complete bipartite. Moreover, in these instances, the average facility opening costs are approximately 15 times higher than the average arc costs.

3.2 Separation Algorithms

It is well known that cut inequalities (yCuts) and (aCuts) can both be separated in polynomial time using a max-flow algorithm (once for each core or customer node, respectively). Since, all coefficients in the objective function are nonnegative, we also obtain a valid model for aConFL when replacing (yCuts) by the following, so-called (zCuts) inequalities:

$$x(\delta^-(H)) \geq z_i \quad \forall H \subseteq S, \forall i \in H \cap I \tag{zCuts}$$

Though (zCuts) are not facet-inducing, they performed well in practice (see [1]).

3.3 Results

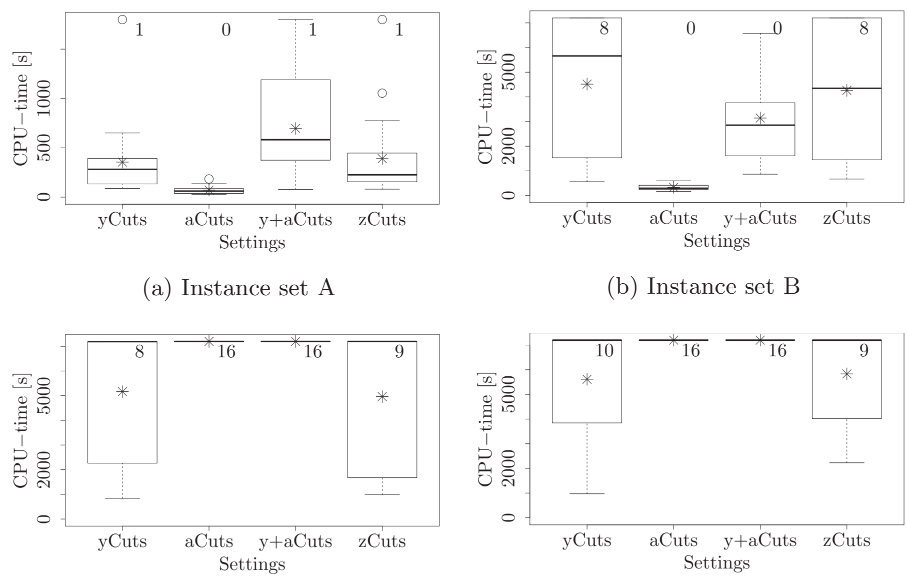
The computational results have been obtained using an Intel Xeon X5500 with 2.67Ghz and 24GB RAM and CPLEX 12.5 as solver for the ILPs. CPLEX-cuts have been turned off and the highest branching priority was given to facility variables. Before starting the solution process, all polynomial-size constraints, plus the inequalities $x_{st} + x_{ts} \leq y_s$, $\forall s \in S$, are added to the model. We have developed a branch-and-cut approach and tested the performance of the following settings: 1) `yCuts`: separating (yCuts) only; 2) `aCuts`: separating (aCuts) only; 3) `y+aCuts`: separating (yCuts), and only if no (yCuts) are violated in a branch-and-bound node, (aCuts) are separated; 4) `zCuts`: separating (zCuts) only. The separation routine for a node s (facility i) is called only if the corresponding LP-value on the right-hand-side is ≥ 0.5.

Figures 1a– 1d show boxplots of the runtimes (in seconds) over all instances from A, B, C and D, respectively. The star in the boxplot indicates the average solution time and the number on top of each plot indicates the number of instances, which could not be solved within the given timelimit (two hours for B-D and 30 minutes for A).

There is a clear contrast in the performance on the instances A, B and instances C, D. For the former ones, the `aCuts`-setting significantly outperforms the remaining setting. On the contrary, the `aCuts`-setting is the worst approach for C, D. This can be explained by the different facility opening costs in the two groups: for A, B instances, opening a facility costs on average as much as establishing a link; however, it is about 15 times as expensive in groups C, D. On average, there are about twelve open facilities in optimal solutions of A, B, while only around four are open for C, D. Consequently, LP-solutions contain much less non-zero z- (and y-) variables in the latter case, and therefore, less separation calls are needed for the `zCuts`- and `yCuts`-setting. On the contrary, the numbers of separation calls for (zCuts) and (aCuts) are comparable for instances A, B. Therefore, the `aCuts`-setting is clearly beneficial, as it implies the strongest LP-bounds (recall that (aCuts) are facet-defining).

Comparing the performance between groups A and B, we observe that the sparsity of instances (group B) seems to deteriorate the performance of `yCuts`- and `zCuts`-settings.

Acknowledgements M. Leitner is supported by the Austrian Science Fund (FWF) under grant I892-N23.



(a) Instance set A

(b) Instance set B

(c) Instance set C

(d) Instance set D

Fig. 1: Runtimes for the different settings

References

1. Gollowitzer, S., Ljubić, I.: MIP models for connected facility location: A theoretical and computational study. Comput. & Oper. Res. **38**(2) (2011) 435–449
2. Leitner, M., Raidl, G.R.: Branch-and-cut-and-price for capacitated connected facility location. J. of Math. Modelling and Algorithms **10**(3) (2011) 245–267
3. Leitner, M., Ljubić, I., Salazar-González, J.J., Sinnl, M.: The connected facility location polytope: Valid inequalities, facets and a computational study. (2014) Submitted to Math. Programming.
4. Fischetti, M.: Facets of two Steiner arborescence polyhedra. Math. Programming **51** (1991) 401–419
5. Cornuejols, G., Thizy, J.M.: Some facets of the simple plant location polytope. Math. Programming **23**(1) (1982) 50–74
6. Nemhauser, G.L., Wolsey, L.A.: Integer and Combinatorial Optimization. Wiley New York (1999)
7. Guignard, M.: Fractional vertices, cuts and facets of the simple plant location problem. In Padberg, M., ed.: Combinatorial Optimization. Volume 12 of Mathematical Programming Studies. Springer Berlin Heidelberg (1980) 150–162
8. Bardossy, M.G., Raghavan, S.: Dual-based local search for the connected facility location and related problems. INFORMS J. Comput. **22**(4) (2010) 584–602
9. Cirasella, J., Johnson, D., McGeoch, L., Zhang, W.: The asymmetric traveling salesman problem: Algorithms, instance generators, and tests. In Buchsbaum, A., Snoeyink, J., eds.: Algorithm Engineering and Experimentation. Volume 2153 of Lecture Notes in Computer Science. Springer Berlin Heidelberg (2001) 32–59