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Stochastic Survivable Network Design Problems: Theory and Practice

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Abstract We study survivable network design problems with edge-connectivity requirements under a two-stage stochastic model with recourse and finitely many scenarios. For the formulation in the natural space of edge variables we show that facet defining inequalities of the underlying polytope can be derived from the deterministic counterparts. Moreover, by using graph orientation properties we introduce stronger cut-based formulations. For solving the proposed MIP models, we suggest a two-stage branch&cut algorithm based on a decomposed model. In order to accelerate the computations, we suggest a new technique for strengthening the decomposed L-shaped optimality cuts which is computationally fast and easy to implement. Computational experiments show the benefit of the decomposition and the cut strengthening which significantly reduces the number of master iterations and the computational running time.

Keywords stochastic network design problems, stochastic integer programming, branch-and-cut, Benders decomposition

1 Introduction

Motivation. Survivable network design problems with edge-connectivity requirements (SNDPs) are among the most fundamental problems in the field of network optimiza-

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tion. Many classical network design problems including the shortest path problem, the minimum spanning tree problem, the Steiner tree problem, the minimum-weight edge-connected subgraph problem, and edge-connectivity augmentation problems are special cases of the survivable network design problem. Applications of the SNDP can be found in many different fields, e.g., in the design of supply chain and distribution networks, in the chip layout design, or in the reconstruction of phylogenetic trees. The field of telecommunications belongs to the most important applications that request building cost-effective networks with higher connectivity requirements.

In a typical telecommunication network [9, 10], an edge weighted graph is given with three *types of nodes*: "special" offices (nodes of type 2), "ordinary" offices (nodes of type 1), and "optional" offices (nodes of type 0). The goal is to find a cost-minimal subnetwork which ensures that all special offices belong to a two-connected component, all ordinary offices are at least simply connected, and optional offices can be used to establish connections, if that would lead to a cheaper solution. The latter problem is known as the $\{0, 1, 2\}$-SNDP. Here, we consider the general SNDP with arbitrary connectivity requirements between each pair of nodes.

In practice, however, from the moment that the information is gathered until the moment in which the solution has to be implemented, some of the data might change with respect to the initial setting. So, for example, even if some (rough) idea about node types is known, changes in demographic, socio-economic, or political factors can lead to changes in the demand requirement of a certain region, or availability of a given location to host an office. Furthermore, the costs of establishing links (installing new pipes, cables, etc.) may be subject to inflations and price deviations. This means that the solution obtained using a classical deterministic method might become suboptimal or even infeasible such that a new solution might have to be redefined from scratch.

Despite the great importance of the SNDP and the relevance of the uncertainty for practical applications, to our knowledge, nothing is known about building survivable networks under data uncertainty. In this article we attempt to close this gap by considering *two-stage stochastic* versions of survivable network design problems with edge- and node-connectivity requirements (for an introduction to stochastic programming see, e.g., [2]). Thereby, the uncertain data is modeled using random variables with a set of scenarios defining their possible outcomes. Typically, a solution is comprised by first- and second-stage decisions such that a partial subnetwork is built in the first stage which is then completed once the uncertain data becomes available (in the second stage).

More precisely, in *the two-stage stochastic survivable network design problem* (SSNDP), network planners want to establish profitable connections now (in the *first stage*, on Monday) while taking all possible outcomes—the *scenarios*—into account. In the future (in the *second stage*, on Tuesday) the actual scenario is revealed, requirements and connection costs are now known, and additional connections can be purchased (through so-called *recourse* actions) to satisfy the now known requirements. The objective is to minimize the *expected costs* of the solution, i.e., the sum of the first stage costs plus the expected costs of the second stage. Thereby, all connectivity requirements for all scenarios have to be satisfied. The formal definition of the SSNDP is given in Section 3.

Previous work. There exists a large body of work on different variants of the deterministic survivable network design problem. We refer to [17, 18] for a comprehensive literature overview on the SNDP. Many polyhedral studies were done in the 90's, see, e.g., [11, 18]. A decade later the question of deriving stronger MIP formulations by orienting the k-connected subgraphs has been considered by e.g. [1, 21]. Among the approximation algorithms for the SNDP, we point to the work of Jain [15] whose approximation factor of two remains the best one up to date.

Regarding the stochastic variants of some simplest variants of the SNDP, there are significantly less results published so far. One of the studied related problems is the two-stage stochastic Steiner tree problem which is a special case in which node types are either zero or one. For this problem both approximation algorithms (see, e.g., [14, 29]) and MIP approaches (see [3]) were developed. For the more general case in which node types are arbitrary natural numbers, up to our knowledge, there only exists an $O(1)$ approximation algorithm (see [13]) for the following special case of the $\{0, k\}$-SSNDP: For each pair of distinct nodes i and j a single scenario, whose probability is p_{ij}, is given in which nodes i and j need to be k-edge-connected. But in general, however, it follows by [26] that the SSNDP is as hard to approximate as label cover—which is $\Omega(\log^{2-\epsilon} n)$ hard. In fact, the hardness-proof already works for the stochastic shortest path problem.

Our contribution. Our contribution is twofold, it concerns theoretical models as well as practical algorithms.

Theory: In the past, the seminal result of Nash and Williams [23] has been used to develop stronger mixed integer programming (MIP) models for the deterministic SNDP by exploiting graph orientations [5, 6, 21]. Here, we show that graph orientation properties cannot be used in a straight-forward fashion to develop similar models for the SSNDP. As an alternative, we propose two general ways to develop *semi-directed* MIP models in which only the second-stage solutions are oriented. We develop two novel cut-based MIP models of the deterministic equivalent for solving the SSNDP on undirected graphs based on these orientation properties. We prove that the new models are stronger than the original one based on standard undirected cuts. Moreover, when considering the undirected formulation of the SSNDP we show that facet defining inequalities can be easily derived from their deterministic counterparts.

Practice: The SSNDP belongs to a broader class of two-stage integer stochastic programs with binary first stage solutions and binary recourse. These NP-hard problems are known to be notoriously difficult to solve [27]. In this paper, we use a recently introduced decomposition approach called *two-stage branch&cut* [3]. This approach uses a Benders decomposition and two nested branch&cut algorithms. In the *subproblems*, violated directed cuts are separated, while the *master problem* is expanded by L-shaped and integer optimality cuts. To enhance the algorithmic performance, we propose a new computationally inexpensive *dual lifting* procedure that strengthens the inserted L-shaped optimality cuts by simple modifications of the dual solutions of the subproblems. To illustrate the effectiveness of the dual lifting procedure, we compare our approach with the classical method by Magnanti and Wong [22] for generating Pareto-optimal L-shaped cuts. Using a large set of realistic instances, we analyze in detail the characteristics of the proposed models and the obtained solutions as well as the performance, behavior, and limitations of the designed algorithmic approach.

Organization of the paper. We start by recalling the basic ILP formulation for the deterministic SNDP in Section 2. Moreover, we summarize the ideas of Magnanti and Raghavan [21] for strengthening the undirected formulation by orienting the solution and describe the corresponding MIP model. This model is the starting point for our models concerning the stochastic SNDP in Section 3. Here, the undirected formulation (Section 3.1) and two stronger semi-directed formulations (Sections 3.2 and 3.3, respectively) are described. Furthermore, structural results for the associated polyhedron with the undirected formulation are provided. Section 4 is dedicated to the two-stage branch&cut algorithm with descriptions of the algorithm and the decomposition. Afterwards, in Section 5, we describe the dual lifting procedure for strengthening the generated L-shaped optimality cuts. The benefit of these cuts—and the decomposition itself—is presented in the results of the experimental study in Section 6.

2 The deterministic survivable network design problem

Definition. Formally, the deterministic version of the SNDP is defined as follows: We are given a simple undirected graph $G = (V, E)$ with edge costs $c_e \geq 0, \forall e \in E$, and a symmetric $|V| \times |V|$ *connectivity requirement* matrix $\mathbf{r} = [r_{uv}]$. Thereby, $r_{uv} \in \mathbb{N} \cup \{0\}$ represents the minimal required number of edge-disjoint paths between two distinct nodes $u, v \in V$. The goal consists of finding a subset of edges $E' \subseteq E$ satisfying the connectivity requirements and minimizing the overall solution costs being defined as $\sum_{e \in E'} c_e$.

While the given definition of the SNDP is as general as possible, one additional assumption is made in this paper which is commonly considered in the literature: it is assumed that the connectivity requirements $\mathbf{r} = [r_{uv}]$ imply that each optimal solution comprises a single connected component. In this case the problem is called the *unitary* SNDP. One example for a *non-unitary* SNDP is the Steiner forest problem where optimal solutions might be disconnected.

A closely related problem to the SNDP is the *node-connectivity SNDP* where requirements have to be satisfied by node-disjoint paths. A small example depicting optimal solutions for both problems is illustrated in Figure 1. For the ease of presentation, this article focusses on edge-connectivity but the ideas and algorithms are transferable to the related node-connectivity problem as well.

We note that in some applications (including our example mentioned in the introduction), edge- or node-connectivity requirements can be specified using node types $\rho_u \in \mathbb{N} \cup \{0\}$, for all $u \in V$. In that case, the required connectivity between a pair of distinct nodes $u, v \in V$ is defined as $r_{uv} = \min\{\rho_u, \rho_v\}$. In presenting the main results of this paper we will stick to the more general definition of connectivity requirements using the connectivity matrix $\mathbf{r}$. For illustrating examples, however, we will use the more intuitive concept of node types.

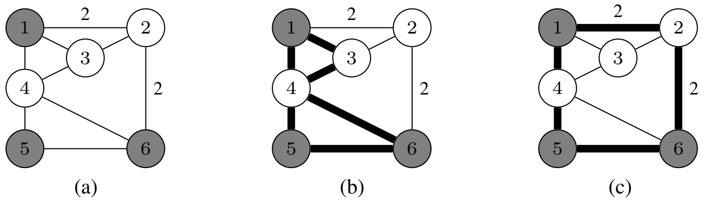


Fig. 1 (a) An example network with gray nodes having connectivity requirement two w.r.t. each other and all other requirements being zero. Default edge costs are one. The bold edges depict the optimal solution for (b) edge-connectivity with costs 6 and (c) node-connectivity with costs 7, respectively.

(M)ILP models. The classical cut-based ILP formulation for the deterministic SNDP (see, e.g., [11,12]) uses binary decision variables x_e for each edge $e \in E$. We use notations $x(E') := \sum_{e \in E'} x_e, \forall E' \subseteq E$, and $\delta(W) := \{e = \{i,j\} \in E \mid |\{i,j\} \cap W| = 1\}$, for $W \subset V$. Moreover, let us denote by

$$f(W) := \max\{r_{uv} \mid u \in W, v \notin W\}, \quad \forall W \subset V,$$

the *connectivity function* on G. The SNDP based on undirected cuts then reads as follows:

$$(SNDP_{ucut}) \ \min \sum_{e \in E} c_e x_e$$

$$s.t. \quad x(\delta(W)) \geq f(W) \qquad \forall \emptyset \neq W \subset V \qquad (ucut{:}1)$$

$$\boldsymbol{x} \in \{0,1\}^{|E|}$$

This model is one of the most famous models in the literature and has been used in polyhedral studies (see, e.g., [11]) or to estimate the quality of approximative solutions (see, e.g., [15]). Magnanti and Raghavan [21] showed how to strengthen this formulation by using a famous theorem by Nash-Williams [23] about graph orientations that we restate here:

Theorem 1 (Nash-Williams [23]) *Let $G = (V, E)$ be an undirected graph and let κ_{uv} be the maximum number of edge-disjoint paths from u to v, where $u, v \in V$, $u \neq v$. Then G has an orientation such that for every pair of nodes u and v in G, the number of pairwise edge-disjoint directed paths from u to v in the resulting directed graph is at least $\lfloor \frac{\kappa_{uv}}{2} \rfloor$.*

If connectivity requirements are in $\{0,1\} \cup \{2k \mid k \in \mathbb{N}\}$ then it is possible to orient any optimal SNDP solution as follows: Since we are dealing with the unitary SNDP, any optimal SNDP solution consists of edge-biconnected components connected with each other by cut nodes or bridges. Using the result of Theorem 1, each of those edge-biconnected components can be oriented such that for each pair of distinct nodes u and v from the same component there exist $r_{uv}/2$ edge-disjoint directed paths from u to v and $r_{uv}/2$ edge-disjoint directed paths from v to u. To orient possible bridges, a node v_r is chosen for which we know that it is a part of an edge-biconnected component and each bridge is oriented away from this component. To this end, the edge-biconnected components are oriented, shrunk into single nodes, and the obtained tree is oriented away from the "root" v_r. These orientation properties can be used to derive a MIP model that uses binary arc variables d_{ij} associated to the orientation. By projecting the arc variables into the space of undirected edges as $x_e = d_{ij} + d_{ji}$, for all $e = \{i, j\} \in E$, it is not difficult to see that the obtained directed model is stronger than the undirected one given above. In fact, the directed model is strictly stronger if and only if there exists a pair of distinct nodes $u, v \in V$, such that $r_{uv} = 1$ [21].

To model the general SNDP—i.e., the SNDP with arbitrary connectivity requirements $r_{uv} \in \mathbb{N} \cup \{0\}$—Magnanti and Raghavan [21] present an *extended MIP formulation* which is similar to the one described above with the only difference that the binary arc variables d_{ij} are relaxed to be continuous. This small change makes the model valid for arbitrary values of r_{uv} and provably stronger than its undirected counterpart. For describing this model, we will need the following notation: let A be the arc set of the *bidirection* of G containing two directed arcs for each undirected edge, i.e., $\forall \{i, j\} \in E : (i, j), (j, i) \in A$. For a vertex set $W \subset V$ let $\delta^-(W) = \{(i, j) \in A \mid i \notin W, j \in W)\}$ and analogously $\delta^+(W) = \{(i, j) \in A \mid i \in W, j \notin W)\}$ be the set of ingoing and outgoing arcs, respectively. By using fractional arc variables $d_{ij}, \forall (i, j) \in A$, the resulting model by [21] reads as follows:

$$(SNDP_{dcut}) \ \min \sum_{e \in E} c_e x_e$$

$$\begin{aligned}
s.t.\quad & d(\delta^-(W)) \geq f(W)/2 && \forall \emptyset \neq W \subset V, f(W) \geq 2 && (dcut{:}1)\\
& d(\delta^-(W)) \geq 1 && \forall \emptyset \neq W \subset V, f(W) = 1, v_r \notin W && (dcut{:}2)\\
& d_{ij} + d_{ji} \leq x_e && \forall e = \{i, j\} \in E && (dcut{:}3)\\
& d_{ij} \geq 0 && \forall (i, j) \in A && (dcut{:}4)\\
& \boldsymbol{x} \in \{0, 1\}^{|E|} && &&
\end{aligned}$$

Constraints $(dcut{:}2)$ are classical directed Steiner cuts implying connectivity of the solution. The directed cuts $(dcut{:}1)$ ensure that for each two distinct nodes $u, v \in V$ such that $u \in W, v \notin W$, $r_{uv}/2$ directed paths are selected from u to v and from v to u, respectively. The capacity constraints $(dcut{:}3)$ enforce that each selected directed arc is payed for in the objective function.

The main results concerning the (strength of the) two presented formulations are summarized in the following theorem. Let $\mathcal{P}_{SNDP_{ucut}}, \mathcal{P}_{SNDP_{dcut}}$ denote the polyhedrons defined by the linear relaxation of $(SNDP_{ucut})$ and $(SNDP_{dcut})$, respectively. Moreover, let $\text{Proj}_x(\mathcal{P}_{SNDP_{dcut}})$ denote the linear projection of $\mathcal{P}_{SNDP_{dcut}}$ onto the space of undirected x-variables.

Theorem 2 (Magnanti-Raghavan [21]) $(SNDP_{dcut})$ *is a valid formulation for the SNDP. Furthermore,* $(SNDP_{dcut})$ *is strictly stronger than* $(SNDP_{ucut})$*, i.e.,* $Proj_x(\mathcal{P}_{SNDP_{dcut}}) \subseteq \mathcal{P}_{SNDP_{ucut}}$ *and there exist instances for which the strict inequality holds.*

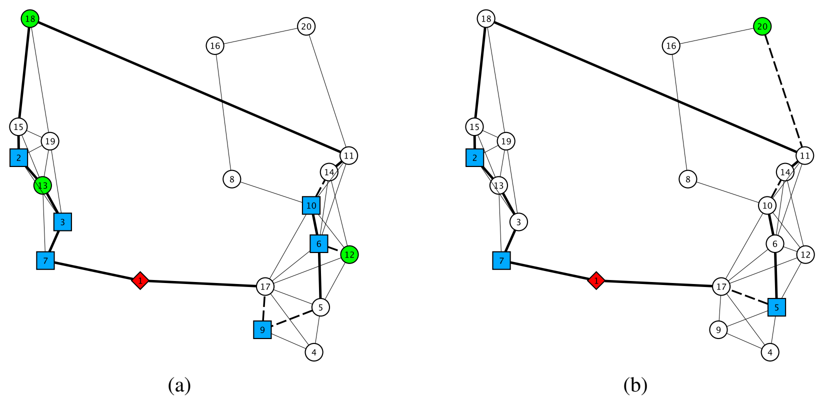


Fig. 2 An example SSNDP instance with 20 nodes, 40 edges, and 2 scenarios. Nodes depicted as blue rectangles imply connectivity requirement 2 and green circles imply connectivity 1. (a) and (b) show the optimal solution for the first and second scenario, respectively, with bold edges being first- and dashed edges being selected second-stage edges.

3 The stochastic survivable network design problem

Definition. Let $G = (V, E)$ denote the undirected input graph with known first-stage edge costs $c^0_e \geq 0$ for all $e \in E$. The actual connectivity requirements as well as the future edge costs are only known in the second stage. These values together form a random variable ξ for which we assume that it has finite support. It can therefore be modeled using a finite set of scenarios $\mathcal{K} = \{1, \dots, K\}$, $K \geq 1$. The realization probability of each scenario is given by $p^k > 0$, $k \in \mathcal{K}$, with $\sum_{k \in \mathcal{K}} p^k = 1$. Edge costs in the second stage under scenario $k \in \mathcal{K}$ are denoted by $c^k_e \geq 0$ for all $e \in E$. Furthermore, let $\mathbf{r}^k$ be the matrix of *unitary* connectivity requirements in the k-th scenario.

The *two-stage stochastic survivable network design problem* (SSNDP) is defined as follows: Determine the subset of edges $E^0 \subseteq E$ to be purchased in the first stage and the sets $E^k \subseteq E$ of additional (*recourse*) edges to be purchased in each scenario $k \in \mathcal{K}$, such that the overall expected costs defined as $\sum_{e \in E^0} c^0_e + \sum_{k \in \mathcal{K}} p^k \sum_{e \in E^k} c^k_e$ are minimized. Thereby, $E^0 \cup E^k$ has to satisfy all connectivity requirements between each pair of nodes defined by $\mathbf{r}^k$ for all $k \in \mathcal{K}$. W.l.o.g. we assume that $\sum_{k \in \mathcal{K}} p^k c^k_e \geq c^0_e$, for all $e \in E$; Otherwise, one would never install such an edge in the first stage. We also assume that the connectivity requirements are *unitary*, as in the deterministic setting, i.e., for each $k \in \mathcal{K}$, the connectivity requirements $\mathbf{r}^k$ imply that any optimal solution $E^0 \cup E^k$ is a connected subgraph. Figure 2 shows an example of an SSNDP instance with $k = 2$ scenarios, and its optimal solution.

Solution Topology: Observe that, despite the fact the connectivity requirements are unitary, the optimal first-stage solution E^0 of the SSNDP is not necessarily connected. This holds even if the connectivity requirements are from the set $\{0, 1\}$ only. The optimal first-stage solution may contain several disjoint components depending on the values r^k_{uv} throughout different scenarios—or depending on the second-stage

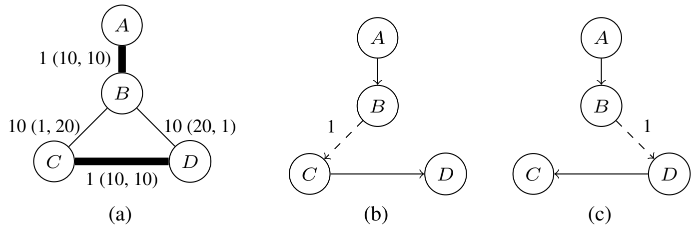


Fig. 3 (a) An instance for the SSNDP with a highlighted optimal first-stage solution being a forest, cf. text. Figures (b) and (c) depict the optimal solution for scenarios 1 and 2, respectively, with selected first stage edges being drawn as solid lines and second stage edges as dashed lines. For the orientations in the semi-directed models we assume vertex A being the root node.

cost structure. A small example is shown in Figure 3: This instance consists of $K = 2$ scenarios with equal probability and with the set of nodes $\{A, C, D\}$ being of "type one" in both scenarios; Node B is of "type zero"in both scenarios. The costs are given next to the edges in Figure 3(a) by "$c_e^0(c_e^1, c_e^2)$", for all edges e. The optimal solution consists of the two edges $\{A, B\}$ and $\{C, D\}$ purchased in the first stage expanded by one edge in scenario 1 (edge $\{B, C\}$, Figure 3(b)) and in scenario 2 (edge $\{B, D\}$, Figure 3(c)). Hence, the overall expected costs are 3.

3.1 Undirected model

We first present the deterministic equivalent—in extensive form—of the SSNDP in the natural space of edge variables. Later on we show how to derive stronger, extended formulations using the orientation properties presented in Section 2 by assigning a unique direction to each edge of a feasible *second stage solution*.

Let binary variables x_e^0 indicate whether an edge $e \in E$ belongs to E^0, and binary second-stage variables x_e^k indicate whether e belongs to E^k, for all scenarios $k \in \mathcal{K}$. For $E' \subseteq E$ let $(x^0 + x^k)(E') := \sum_{e \in E'}(x_e^0 + x_e^k)$. Moreover, we expand the connectivity function to f^k, for each scenario $k \in \mathcal{K}$ and $W \subset V$:

$$f^k(W) := \max\{r_{uv}^k \mid u \notin W, v \in W\}.$$

A deterministic equivalent of the SSNDP can then be modeled easily using undirected cuts as follows:

$$(UD)\ \min \sum_{e \in E} c_e^0 x_e^0 + \sum_{k \in \mathcal{K}} p^k \sum_{e \in E} c_e^k x_e^k$$

$$s.t.\quad (x^0 + x^k)(\delta(W)) \geq f^k(W) \qquad \forall \emptyset \neq W \subset V, \forall k \in \mathcal{K} \qquad (UD{:}1)$$

$$x_e^0 + x_e^k \leq 1 \qquad \forall e \in E, \forall k \in \mathcal{K} \qquad (UD{:}2)$$

$$(\boldsymbol{x^0}, \boldsymbol{x^1}, \ldots, \boldsymbol{x^K}) \in \{0,1\}^{|E|(K+1)}$$

This model is a direct extension of the model from Section 2. Constraints (UD:1) ensure edge-connectivity between each pair of nodes in each scenario realization while first- and second-stage edges can be used. The additional constraints (UD:2) simply forbid the installation of the same edge in the first-stage and in any scenario.

Polyhedral properties. Let $\mathcal{S}^k$ be the convex hull of all integer points that define feasible SNDP solutions w.r.t. connectivity requirements $\mathbf{r}^k$, i.e.,

$$\mathcal{S}^k = \text{conv}\{\boldsymbol{x^k} \in \{0,1\}^{|E|} \mid x^k(\delta(W)) \geq f^k(W), \forall \emptyset \neq W \subset V\}.$$

Similarly, let $\mathcal{S}$ be the convex hull of all integer points that define feasible SSNDP solutions, i.e.,

$$\mathcal{S} = \text{conv}\{\boldsymbol{x} = (\boldsymbol{x^0}, \boldsymbol{x^1}, \ldots, \boldsymbol{x^K}) \in \{0,1\}^{|E|(K+1)} \mid \boldsymbol{x} \text{ satisfies } (UD{:}1),(UD{:}2)\}.$$

In the following, we will study some properties of the polytope $\mathcal{S}$.

Lemma 1 *If the polytopes $\mathcal{S}^k$ for all $k \in \mathcal{K}$ are full-dimensional, then the polytope $\mathcal{S}$ is full-dimensional as well.*

Proof Let $m := |E|$. We need to show that $\dim(\mathcal{S}) = m(K+1)$. To this end, we now construct a matrix that contains $mK + m$ linearly independent feasible solutions to SSNDP. In the last step we extend it by one more solution with the whole collection of solutions being affinely independent.

The matrix is constructed in $(K+1)\cdot(K+1)$ blocks of size $m \times m$ and each row of the matrix represents one feasible solution in $\mathcal{S}$. Each block column corresponds to a binary variable of the vector $(\boldsymbol{x^0}, \boldsymbol{x^1}, \ldots, \boldsymbol{x^K})$; the first m rows represent feasible independent solutions involving the $\boldsymbol{x^0}$ variables, the next Km solutions are linearly independent with respect to the $\boldsymbol{x^k}$ variables, for all $k \in \mathcal{K}$.

We first observe that for each scenario $k \in \mathcal{K}$, the collection of m solutions $\mathcal{E} = \{E \setminus \{e\} \mid e \in E\}$ represents a set of m linearly independent points of the polytope $\mathcal{S}^k$. Let $\boldsymbol{A_0}$ denote the $m \times m$ matrix obtained by row-wise concatenation of the characteristic vectors of these solutions.

1. Initialize the first $m \times m$ block with $\boldsymbol{A_0}$. Fill out the remaining K blocks at position $[0,k], k \in \mathcal{K}$, with the $m \times m$ identity matrix $\boldsymbol{I_m}$.
2. For $k \in \mathcal{K}$: set up the block at the position $[k,0]$ to $\mathbf{0}$, and the block at the position $[k,k]$ to $\boldsymbol{A_0}$. The remaining blocks at positions $[\ell,k]$ are set to $\mathbf{1}$, for all $\ell \in \mathcal{K}, \ell \neq k$.

It is not difficult to see that the obtained matrix has full rank $mK + m$. In the last step, we add the vector that is obtained by concatenating the $\mathbf{0}$ vector solution for $\boldsymbol{x^0}$ and $\mathbf{1}$'s for the remaining coordinates $\boldsymbol{x^1}$ to $\boldsymbol{x^K}$. Subtracting all solutions contained in the matrix from the latter solution gives a new matrix—with full rank, too. Hence, all solutions are affinely independent. □

Theorem 3 *If for all $k \in \mathcal{K}$, the polytopes $\mathcal{S}^k$ are full-dimensional and the inequality $\boldsymbol{\pi x^\ell} \geq \pi_0$, with $\pi_e \in \mathbb{N} \cup \{0\}, \forall e \in E$, and $\pi_0 \geq 1$, defines a facet of the polytope $\mathcal{S}^\ell$ for some $\ell \in \mathcal{K}$, then the inequality $\boldsymbol{\pi x^0} + \boldsymbol{\pi x^\ell} \geq \pi_0$ is facet defining for the SSNDP polytope $\mathcal{S}$.*

Proof We denote the affine independent solutions of the polytope $\mathcal{S}^\ell$ that satisfy $\boldsymbol{\pi x^\ell} = \pi_0$ by $T_1^\ell, \ldots, T_m^\ell$. Since $\pi_0 \geq 1$, these points are also linearly independent (the origin does not belong to the set of feasible points). Let $\boldsymbol{A^\ell}$ be the matrix obtained by the row-wise insertion of these solutions, and let $\mathbf{1} - \boldsymbol{A^\ell}$ be the complementary matrix of $\boldsymbol{A^\ell}$. Construction is done by a row-wise insertion of blocks, similarly as above.

1. Initialize the first $m \times m$ block with $\boldsymbol{A^\ell}$ (meaning, set first stage solutions to be equal to the solutions of $\mathcal{S}^\ell$). Fill out the remaining blocks at the position $[0, k]$ with $\mathbf{1} - \boldsymbol{A^\ell}$, for all $k \in \mathcal{K}$, $k \neq \ell$. The block at the position $[0, \ell]$ is set to $\mathbf{0}$.
2. For $k \in \mathcal{K}$, $k \neq \ell$: set up the block at the position $[k, 0]$ to $\mathbf{0}$, at the position $[k, \ell]$ to $\boldsymbol{A^\ell}$ and the block at the position $[k, k]$ to $\boldsymbol{A_0}$ (defined above). The remaining blocks at $[k, i]$ are set to $\mathbf{1}$, for all $i \in \mathcal{K}, i \neq k, \ell$.
3. Set up the block at the position $[\ell, 0]$ to $\mathbf{0}$, and the block at $[\ell, \ell]$ to $\boldsymbol{A^\ell}$. The remaining blocks at $[\ell, i]$ are set to $\mathbf{1}$, for all $i \in \mathcal{K}, i \neq \ell$.

It is not difficult to see that the obtained matrix has full rank, i.e., $(K + 1)m$, and each row satisfies $\boldsymbol{\pi x^0} + \boldsymbol{\pi x^\ell} = \pi_0$ which concludes the proof. □

Many facet-defining inequalities (see, e.g., [31]), known for the deterministic case, can therefore be easily translated into the facets of the SSNDP. There is a large body of work on polyhedral studies for many variants of the SNDP (see, e.g., surveys in [17, 18]). As an illustration of the obtained result, we will provide here two examples. Facets of the SNDP are usually shown for connectivity requirements given as node types (described above). In the SSNDP context, node types are then defined as ρ_u^k for each scenario $k \in \mathcal{K}$ and each node $u \in V$, and the connectivity requirements would be $r_{uv}^k := \min\{\rho_u^k, \rho_v^k\}$.

For example, for a given SSNDP instance with the node types $\rho^k \in \{0, \dots, \kappa\}$ and a given $k \in \mathcal{K}$, under some special conditions (given in [31]), the undirected cut-set inequalities $(x^0 + x^k)(\delta(W)) \geq f^k(W)$ are facet defining.

Similarly, let $W_1, \dots, W_p \subset V, p > 1$ be a proper partition of V, $[W_1, \dots, W_p]$ denoting the edges having their nodes in two different sets, and $I_1 := \{i \mid f^k(W_i) = 1\}$ and $I_2 := \{i \mid f^k(W_i) > 1\}$. The *partition inequalities*

$$(x^0 + x^k)([W_1, \dots, W_p]) \geq \begin{cases} p - 1 & \text{if } I_2 = \emptyset \\ \lceil 1/2 \sum_{i \in I_2} f^k(W_i) + |I_1| \rceil & \text{otherwise} \end{cases}$$

are facet-defining for $\mathcal{S}$ (see [31] for the necessary and sufficient conditions).

Let F be an extended MIP formulation for the SSNDP. We will denote the polyhedron defined by the LP-relaxation of F by $\mathcal{P}_F$ and the natural projection of $\mathcal{P}_F$ onto the space of undirected $(\boldsymbol{x^0}, \boldsymbol{x^1}, \dots, \boldsymbol{x^K})$ variables by $\text{Proj}_{(x^0,\dots,x^K)}(\mathcal{P}_F)$. Furthermore, for two (extended) formulations F_1 and F_2, we will say that F_1 is *strictly stronger* than F_2 if and only if $\text{Proj}_{(x^0,\dots,x^K)}(\mathcal{P}_{F_1}) \subsetneq \text{Proj}_{(x^0,\dots,x^K)}(\mathcal{P}_{F_2})$ and there exists an SSNDP instance for which the strict inequality holds and for which the bound of the LP-relaxation of F_1 is tighter than the one of the LP-relaxation of F_2.

3.2 Semi-directed model

It is known that MIP models on bidirected graphs provide better LP-based lower bounds for many types of network design problems, cf. Section 2. Therefore we are looking for a possibility to strengthen the model (UD) by bi-directing the given graph G and replacing edge- by arc-variables in the same model. The main difficulty with the SSNDP arises from the fact that the first-stage solution may be disconnected (cf.

Figure 3), and as such it cannot be oriented, even though the deterministic counterpart admits an orientation. In the following, we will introduce two extended formulations (semi-directed models) to overcome these obstacles and provide two MIP models that are strictly stronger than the model (UD).

The example given in Figure 3 illustrates an instance in which it is impossible to uniquely orient the edges from E^0 since there exists an edge from E^0 that cannot be used in exactly the same direction over all scenarios. More precisely, the edge $\{C, D\}$ is selected in the first stage and during the orientation process it is used in the direction (C, D) in scenario 1 and in the direction (D, C) in scenario 2, respectively. Therefore, requiring a fixed orientation for an edge in the first stage would conflict with optimal scenario solutions and would in total lead to more expensive, suboptimal solutions.

Hence, the first stage decision variables need to remain associated with undirected edges. However, one can provide a directed formulation once the solution gets completed in the second stage, i.e., one can orient the edges of $E^0 \cup E^k$ independently for each scenario, as depicted in Figures 3(b) and 3(c). We set the root v_r^k for each scenario $k \in \mathcal{K}$ to be one of the nodes with the highest connectivity requirement and search for individual orientations of the scenario solutions $E^0 \cup E^k$, for each $k \in \mathcal{K}$.

By borrowing the notation from [1], let

$$\begin{aligned}
\mathcal{W}_1^k :=& \{W \mid W \subset V, \max\{r_{uv}^k \mid u \notin W, v \in W\} = 1, v_r^k \notin W\} \\
\mathcal{W}_{\geq 2}^k :=& \{W \mid W \subset V, \max\{r_{uv}^k \mid u \notin W, v \in W\} \geq 2\}
\end{aligned}$$

be the set of *regular cutsets* and *critical cutsets*, respectively.

Given the installation of undirected edges from the first stage, the following model constructs oriented second stage solutions. As above, we use variables $\boldsymbol{x^0}, \ldots, \boldsymbol{x^K}$ to model the solution edges. In addition, we introduce continuous variables $\boldsymbol{d^1}, \ldots, \boldsymbol{d^K}$ associated to directed arcs to "orient" the second stage solutions. The first semi-directed model is called (SD_1):

$$(SD_1)\ \min \sum_{e \in E} c_e^0 x_e^0 + \sum_{k \in \mathcal{K}} p^k \sum_{e \in E} c_e^k x_e^k$$

$$\begin{aligned}
s.t. \quad (x^0 + x^k)(\delta(W)) &\geq f^k(W) && \forall W \in \mathcal{W}_{\geq 2}^k, \forall k \in \mathcal{K} && (SD_1{:}1)\\
x^0(\delta(W)) + d^k(\delta^-(W)) &\geq 1 && \forall W \in \mathcal{W}_1^k, \forall k \in \mathcal{K} && (SD_1{:}2)\\
d_{ij}^k + d_{ji}^k &\leq x_e^k && \forall e = \{i,j\} \in E, \forall k \in \mathcal{K} && (SD_1{:}3)\\
x_e^0 + x_e^k &\leq 1 && \forall e \in E, \forall k \in \mathcal{K} && (SD_1{:}4)\\
d_{ij}^k &\geq 0 && \forall (i,j) \in A, \forall k \in \mathcal{K} && (SD_1{:}5)\\
(\boldsymbol{x^0}, \boldsymbol{x^1}, \ldots, \boldsymbol{x^K}) &\in \{0,1\}^{|E|(K+1)}
\end{aligned}$$

Constraints $(SD_1{:}1)$ ensure that in each scenario k there are at least r_{uv}^k edge-disjoint paths between u and v, $u \in W, v \notin W$, consisting of first- and second-stage edges. Due to constraints $(SD_1{:}2)$ there is at least one path from the root node v_r^k to each vertex u whenever $r_{v_r^k u} = 1$. If an edge is purchased in the second stage, then constraints $(SD_1{:}2)$ associated to bridges will force the orientation of those bridges away from the root node v_r^k. Furthermore, since variables d_{ij}^k are fractional, by using the same arguments as in Theorem 2 the model is valid for any $r_{uv}^k \in \mathbb{N} \cup \{0\}$. Hence, we have the following lemma.

Lemma 2 *Formulation* (SD_1) *models the deterministic equivalent of the two-stage stochastic survivable network design problem correctly.*

Proof Consider an optimal solution to an SSNDP instance with its characteristic binary vectors being described by $\tilde{\boldsymbol{x}} := (\tilde{\boldsymbol{x}}^{\boldsymbol{0}}, \tilde{\boldsymbol{x}}^{\boldsymbol{1}}, \dots, \tilde{\boldsymbol{x}}^{\boldsymbol{K}})$. Using $\tilde{\boldsymbol{x}}$ as solution to (SD_1), all undirected cuts (constraints $(SD_1$:1)) are obviously satisfied. Moreover, due to the results of [23] and [21], it is possible to find an orientation in each scenario such that constraints $(SD_1$:2) are satisfied. Following this orientation the values for the directed variables d^k can be set accordingly. Non-negativity and all other constraints follow directly. Hence, there is a feasible solution to (SD_1) with the same objective value.

On the other hand, each optimal solution to (SD_1) obviously satisfies all connectivity requirements and induces a solution to the SSNDP with the same costs. □

Note that constraints $(SD_1$:1) can also be expressed as $x^0(\delta(W)) + d^k(\delta^-(W)) + d^k(\delta^+(W)) \geq f^k(W)$ (which better explains the original intention of this model). However, one easily observes that this is just an equivalent way of rewriting $(SD_1$:1), without any influence on the lower bounds of the given model.

Theorem 4 *The semi-directed formulation* (SD_1) *is strictly stronger than the undirected formulation* (UD).

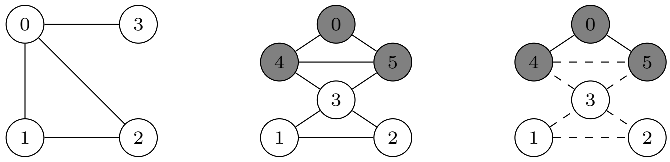
Proof It follows from the proof of Lemma 2 that any solution $(\tilde{\boldsymbol{x}}^{\boldsymbol{0}}, \tilde{\boldsymbol{x}}^{\boldsymbol{1}}, \dots, \tilde{\boldsymbol{x}}^{\boldsymbol{K}}) \in \text{Proj}_{(x^0,\dots,x^K)}(\mathcal{P}_{SD_1})$ is a valid solution to (UD) with the same objective value.

The strict inequality concerning the strength of the formulations is shown by the example given in Figure 4(a). Assume there are 2 scenarios with equal probability and non-zero connectivity requirements $r_{01}^1 = r_{02}^1 = r_{12}^1 = 1$, $r_{03}^2 = 1$, and $c_e^0 = 10, c_e^k = 12, \forall e \in E, k \in \{1,2\}$. In the optimal solution of (UD), only edges in the second stage are purchased with a total objective value of 15: $x_{01}^1 = x_{12}^1 = x_{02}^1 = 0.5$ and $x_{03}^2 = 1$. On the other hand, this solution is infeasible for the relaxed model of (SD_1), i.e., there is no solution to (SD_1) with the same objective value. □

3.3 Stronger semi-directed formulation

In the following model, which represents an alternative model to (SD_1), binary edge variables $\boldsymbol{y^k}$ are used to model the second-stage solution. These variables additionally include the edges that are already bought in the first stage, i.e., we have $y_e^k = 1$ if $e \in E^0 \cup E^k$, and $y_e^k = 0$, otherwise. Moreover, continuous variables z_{ij}^k are used to orient the edges from $E^0 \cup E^k$. The model will be called (SD_2):

$$(SD_2)\ \min \sum_{e \in E} c_e^0 x_e^0 + \sum_{k \in \mathcal{K}} p^k \sum_{e \in E} c_e^k (y_e^k - x_e^0)$$



(a) (b) (c)

Fig. 4 Two counterexamples that prove the strength of the new formulations. (a) Instance with $LP(SD_1) > LP(UD)$, (b) instance with $LP(SD_2) > LP(SD_1)$, and (c) the optimal LP-solution of (SD_1): A solid line represents an LP value of 1, a dashed line a value of 0.5. Gray nodes have connectivity requirement two, all other nodes connectivity requirement one.

$$
\begin{aligned}
s.t.\quad & z^k(\delta^-(W)) \geq f^k(W)/2 && \forall W \in \mathcal{W}^k_{\geq 2}, \forall k \in \mathcal{K} && (SD_2{:}1)\\
& z^k(\delta^-(W)) \geq 1 && \forall W \in \mathcal{W}^k_1, \forall k \in \mathcal{K} && (SD_2{:}2)\\
& z^k_{ij} + z^k_{ji} \geq x^0_e && \forall e = \{i,j\} \in E, \forall k \in \mathcal{K} && (SD_2{:}3)\\
& z^k_{ij} + z^k_{ji} \leq y^k_e && \forall e = \{i,j\} \in E, \forall k \in \mathcal{K} && (SD_2{:}4)\\
& z^k_{ij} \geq 0 && \forall (i,j) \in A, \forall k \in \mathcal{K} && (SD_2{:}5)\\
& (\boldsymbol{x^0}, \boldsymbol{y^1}, \ldots, \boldsymbol{y^K}) \in \{0,1\}^{|E|(K+1)}
\end{aligned}
$$

The directed cuts $(SD_2{:}1)$ and $(SD_2{:}2)$ model the orientation of the solution and ensure the required connectivities independently for each scenario. Notice that due to the symmetry, if $W \in \mathcal{W}^k_{\geq 2}$ it follows that $V \setminus W \in \mathcal{W}^k_{\geq 2}$, too. Hence, for each $W \in \mathcal{W}^k_{\geq 2}$ the ingoing and outgoing cut, i.e., $z^k(\delta^-(W)) \geq f^k(W)/2$ and $z^k(\delta^+(W)) \geq f^k(W)/2$, are contained in (SD_2).

Constraints $(SD_2{:}3)$ and $(SD_2{:}4)$ ensure that variables z^k_{ij} can be used only along the edges that are either purchased in the first stage or added in the second stage. In particular, $(SD_2{:}3)$ forces the orientation of selected first stage edges in each scenario. Therefore, it holds $y^k_e \geq x^0_e$ which explains the corrective term in the objective function. Since the variables z^k_{ij} are fractional, the model is valid for any $r^k_{uv} \in \mathbb{N} \cup \{0\}$:

Lemma 3 *Formulation (SD_2) models the deterministic equivalent of the two-stage stochastic survivable network design problem correctly.*

Proof Again, let $(\tilde{\boldsymbol{x}}^{\boldsymbol{0}}, \tilde{\boldsymbol{x}}^{\boldsymbol{1}}, \ldots, \tilde{\boldsymbol{x}}^{\boldsymbol{K}})$ describe an optimal solution to an SSNDP instance. Following the ideas of the proof of Lemma 2 it is possible to find an orientation for the directed variables z^k using only edges $\{e \in E \mid \tilde{x}^0_e + \tilde{x}^k_e = 1\}$ for each scenario and hence create a valid solution to (SD_2) with the same objective value.

Conversely, an optimal solution $(\bar{\boldsymbol{x}}^{\boldsymbol{0}}, \bar{\boldsymbol{y}}^{\boldsymbol{1}}, \ldots, \bar{\boldsymbol{y}}^{\boldsymbol{K}}, \bar{\boldsymbol{z}}^{\boldsymbol{1}}, \ldots, \bar{\boldsymbol{z}}^{\boldsymbol{K}})$ to (SD_2) satisfies all edge connectivity requirements in each scenario due to the correctness of the directed formulation $(SNDP_{dcut})$. Thereby, capacity constraints $(SD_2{:}3)$ imply—analogously to the deterministic case—that $(\bar{\boldsymbol{x}}^{\boldsymbol{0}}, (\boldsymbol{x^k} := \bar{\boldsymbol{x}}^{\boldsymbol{0}} - \bar{\boldsymbol{y}}^{\boldsymbol{k}})_{k=1,\ldots,K})$ is a feasible solution to the SSNDP. □

Theorem 5 *The semi-directed formulation (SD_2) is strictly stronger than the semi-directed formulation (SD_1).*

Proof Let $(\hat{\boldsymbol{x}}^{\boldsymbol{0}}, \hat{\boldsymbol{y}}^{\boldsymbol{1}}, \dots, \hat{\boldsymbol{y}}^{\boldsymbol{K}}, \hat{\boldsymbol{z}}^{\boldsymbol{1}}, \dots, \hat{\boldsymbol{z}}^{\boldsymbol{K}}) \in \mathcal{P}_{SD_2}$. For $k \in \mathcal{K}$ and $(i,j) \in A$ set $\lambda_{ij}^k := 0$ if $z_{ij}^k + z_{ji}^k = 0$ and $\lambda_{ij}^k := z_{ij}^k/(z_{ij}^k + z_{ji}^k)$, otherwise. Hence, $\lambda_{ij}^k + \lambda_{ji}^k = 1, \forall \{i,j\} \in E$ and $k \in \mathcal{K}$ with $\hat{z}_{ij}^k + \hat{z}_{ji}^k > 0$. Moreover, set $\bar{\boldsymbol{x}}^{\boldsymbol{0}} := \hat{\boldsymbol{x}}^{\boldsymbol{0}}, \bar{\boldsymbol{x}}^{\boldsymbol{k}} := \hat{\boldsymbol{y}}^{\boldsymbol{k}} - \hat{\boldsymbol{x}}^{\boldsymbol{0}}$, and $\forall (i,j) \in A$ with $e = \{i,j\} : \bar{d}_{ij}^k := \hat{z}_{ij}^k - \lambda_{ij}^k \hat{x}_e^0$, for all $k \in \mathcal{K}$.

Obviously, interpreting $(\bar{\boldsymbol{x}}^{\boldsymbol{0}}, \bar{\boldsymbol{x}}^{\boldsymbol{1}} \dots, \bar{\boldsymbol{x}}^{\boldsymbol{K}}, \bar{\boldsymbol{d}}^{\boldsymbol{1}}, \dots, \bar{\boldsymbol{d}}^{\boldsymbol{K}})$ as (SD_1)-solution gives the same objective value. This solution is also feasible due to the following arguments.

The connectivity constraints (SD_1:1) are satisfied since for each $W \in \mathcal{W}_{\geq 2}^k$, $k \in \mathcal{K}$, we have:

$$
\begin{aligned}
&(\bar{x}^0 + \bar{x}^k)(\delta(W)) = \hat{x}^0(\delta(W)) + \hat{y}^k(\delta(W)) - \hat{x}^0(\delta(W)) \\
\geq &\sum_{e=\{i,j\} \in \delta(W)} (\hat{z}_{ij}^k + \hat{z}_{ji}^k) = \hat{z}^k(\delta^-(W)) + \hat{z}^k(\delta^+(W)) \geq f^k(W)
\end{aligned}
$$

The 1-connectivity constraints (SD_1:2) are also fulfilled since for each $W \in \mathcal{W}_1^k$, $k \in \mathcal{K}$, we have:

$$
\begin{aligned}
\bar{x}^0(\delta(W)) + \bar{d}^k(\delta^-(W)) &= \hat{x}^0(\delta(W)) + \sum_{(i,j) \in \delta^-(W), e=\{i,j\}} (\hat{z}_{ij}^k - \lambda_{ij}^k \hat{x}_e^0) \\
&\geq \hat{z}^k(\delta^-(W)) \geq 1
\end{aligned}
$$

The remaining constraints are also satisfied: (SD_1:3): $\bar{d}_{ij}^k + \bar{d}_{ji}^k = \hat{z}_{ij}^k + \hat{z}_{ji}^k - \hat{x}_e^0 \leq \hat{y}_e^k - \hat{x}_e^0 = \bar{x}_e^k$, ($SD_1$:4): $\bar{x}_e^0 + \bar{x}_e^k = \hat{x}_e^0 + \hat{y}_e^k - \hat{x}_e^0 \leq 1$. Moreover, d_{ij}^k-variables are non-negative: $\bar{d}_{ij}^k = \hat{z}_{ij}^k - (\hat{z}_{ij}^k/(\hat{z}_{ij}^k + \hat{z}_{ji}^k))\hat{x}_e^0 \geq \hat{z}_{ij}^k - (\hat{z}_{ij}^k/(\hat{z}_{ij}^k + \hat{z}_{ji}^k))(\hat{z}_{ij}^k + \hat{z}_{ji}^k) = 0$.

Last but not least, it trivially holds $\bar{\boldsymbol{x}}^{\boldsymbol{0}} \in [0,1]^{|E|}$ and $\bar{\boldsymbol{x}}^{\boldsymbol{k}} \in [0,1]^{|E|}, \forall k \in \mathcal{K}$, since $\bar{x}_e^k = \hat{y}_e^k - \hat{x}_e^0$ and $\hat{y}_e^k \geq \hat{x}_e^0$. Hence, $(\bar{\boldsymbol{x}}^{\boldsymbol{0}}, \bar{\boldsymbol{x}}^{\boldsymbol{1}} \dots, \bar{\boldsymbol{x}}^{\boldsymbol{K}}, \bar{\boldsymbol{d}}^{\boldsymbol{1}}, \dots, \bar{\boldsymbol{d}}^{\boldsymbol{K}})$ is a feasible solution for (SD_1) with the same objective value.

To show that there exists an instance for which the strict inequality holds, consider the graph shown in Figure 4(b). We assume that the input consists of a single scenario in which the gray nodes require two-connectivity and the remaining ones only one-connectivity. Furthermore, all edge costs are 1 in the first stage and 10 in the second stage. The LP solution shown in Figure 4(c) shows the first-stage solution—nothing needs to be purchased in the second stage—with a total objective value of 5. This solution is valid for the model (SD_1) but it is impossible to "orient" this solution such that it becomes feasible for the model (SD_2). □

Intuitively, formulation (SD_2) gives stronger lower bounds than (SD_1) for instances where the (fractional) undirected first stage edges allow no valid orientation. Here, the first stage covers cuts in (SD_1) but (SD_2) has to purchase additional arcs in the second stage to ensure feasibility.

4 Decomposition

Notice that, even for a constant number of scenarios, our models contain an exponential number of constraints associated with directed cuts. Although these cuts can be separated in a cutting-plane fashion leading to a branch&cut approach, the main drawback of such a *single-stage* branch&cut approach is that we still have to deal with a large set of variables; e.g., formulation (SD_2) contains $|E|(3K+1)$ variables.

Alternatively, one may consider a traditional implementation of Benders decomposition for two-stage stochastic integer problems. A drawback of the latter approach is the need for solving several MIP problems (master problem and subproblems) at each iteration in order to obtain a single Benders cut. Nonetheless, nowadays most of MIP solvers provide branch&cut frameworks such that Benders decomposition can be implemented as a pure branch&cut approach by the use of callbacks. In the stochastic programming context, Benders cuts are added to the master problem to model valid lower-bounds on the expected second stage costs. This idea has been also exploited in various other applications (including some deterministic problems) where "complicated variables" are projected out and replaced by Benders cuts [4,20]. Typically, finding a single violated Benders cut requires solving several *compact MIP or LP models*.

When applying the Benders decomposition concept to the proposed MIP models for the SSNDP, the main difficulty arises with the fact that the associated subproblems contain an exponential number of constraints, and can therefore be solved only by means of a cutting plane approach (for finding optimal LP solutions), or branch&cut (for finding optimal integer solution). Therefore, in order to apply a Benders-like decomposition, one needs to *nest branch&cut algorithms*: a branch&cut is employed for solving the master problem and violated Benders cuts are detected by solving K other branch&cuts associated to the K subproblems. In [3], this approach has been coined *two-stage branch&cut algorithm (2-stage b&c)*.

More precisely, in the 2-stage b&c, the variables of the first stage are kept in the master problem, and the second stage variables are projected out and replaced by a single variable per scenario, i.e., $\Theta^k, \forall k \in \mathcal{K}$. The objective function of the decomposed model becomes $\min \sum_{e \in E} c_e^0 x_e^0 + \Theta$ with the expected second stage costs $\Theta = \sum_{k \in \mathcal{K}} p^k \Theta^k$ and with Θ^k representing a lower bound on the value of the second stage subproblem in scenario $k \in \mathcal{K}$. For a fixed first stage decision $\tilde{\boldsymbol{x}}^{\mathbf{0}}$, the problem decomposes into K smaller subproblems, each of which can be independently solved using a branch&cut approach. Dual variables of the LP-relaxations of these subproblems impose L-shaped cuts that are added to the master while the exact solutions of the subproblems impose integer optimality cuts [19,32]. Computational results of [3], applied to the stochastic Steiner tree problem, have shown that the 2-stage b&c significantly outperforms the branch&cut applied directly to the deterministic equivalent in the extended formulation.

The key results that allow us to apply the 2-stage b&c to the SSNDP and detect violated L-shaped cuts in polynomial time are: 1) the fact that all considered cut inequalities can be separated in polynomial time (see, e.g., [31]), and 2) the famous result by Grötschel, Lovász, and Schrijver (cf., e.g, [24]) that shows the equivalence of optimization and separation—if one can solve the separation problem in polynomial time, then the underlying LP-problem can also be solved in polynomial time. Consequently, in each of the subproblems, the number of tight inequalities of an arbitrary LP optimal solution is polynomial, and therefore, only a polynomial number of dual multipliers will be non-zero in the associated L-shaped constraint. Therefore, we are able to apply the 2-stage b&c to any of the three formulations considered in this paper. For the ease of presentation, we will demonstrate how to solve the SSNDP using the 2-stage b&c applied to the strongest of the three models, namely (SD_2). The main details of this algorithm are provided below in Section 4.2.

4.1 Decomposition of the (SD_2) model

For each fixed—and possibly fractional—first-stage solution $\tilde{\boldsymbol{x}}^{\mathbf{0}}$, the second-stage problem decomposes into K independent subproblems, which we will refer to as *restricted deterministic SNDP's*. They are special cases of the deterministic SNDP due to the capacity constraints (P:2). To simplify the notation, we define $\mathcal{W}^k := \mathcal{W}_1^k \cup \mathcal{W}_{\geq 2}^k$ and merge the constraints (SD_2:1) and (SD_2:2) by using functions $\Phi^k : 2^V \mapsto \mathbb{N} \cup \{0\}$, for all $k \in \mathcal{K}$, that give the correct right-hand side of the directed cuts:

$$\Phi^k(W) := \begin{cases} \frac{1}{2}\max\{r_{uv}^k \mid u \notin W, v \in W\}, & W \in \mathcal{W}_{\geq 2}^k \\ 1, & W \in \mathcal{W}_1^k \end{cases}$$

For a given first stage solution $\tilde{\boldsymbol{x}}^{\mathbf{0}}$, and for each $k \in \mathcal{K}$, these subproblems—already transformed into standard form—are given as follows:

$$(P{:}SD_2)\ \min \sum_{e \in E} c_e^k (y_e^k - \tilde{x}_e^0)$$

$$\begin{aligned} s.t.\quad z^k(\delta^-(W)) &\geq \Phi^k(W) && \forall W \in \mathcal{W}^k && (P{:}1)\\ z_{ij}^k + z_{ji}^k &\geq \tilde{x}_e^0 && \forall e = \{i,j\} \in E && (P{:}2)\\ y_e^k - z_{ij}^k - z_{ji}^k &\geq 0 && \forall e = \{i,j\} \in E && (P{:}3)\\ -y_e^k &\geq -1 && \forall e \in E && (P{:}4)\\ z_{ij}^k &\geq 0 && \forall (i,j) \in A && (P{:}5)\\ \boldsymbol{y^k} &\in \{0,1\}^{|E|} \end{aligned}$$

By relaxing the integrality constraints and using dual variables α_W, β_e, γ_e, and τ_e associated to constraints (P:1), (P:2), (P:3), and (P:4), respectively, we obtain the following dual problem, for each scenario $k \in \mathcal{K}$ and fixed first stage solution $\tilde{\boldsymbol{x}}^{\mathbf{0}}$:

$$(D{:}SD_2)\ \max \sum_{W \in \mathcal{W}^k} \Phi^k(W)\alpha_W + \sum_{e \in E} (\tilde{x}_e^0 \beta_e - \tau_e - c_e^k \tilde{x}_e^0)$$

$$\begin{aligned} \gamma_e - \tau_e &\leq c_e^k && \forall e \in E && (D{:}1)\\ \sum_{\substack{W \in \mathcal{W}^k:\\ (i,j) \in \delta^-(W)}} \alpha_W + \beta_e - \gamma_e &\leq 0 && \forall (i,j) \in A && (D{:}2)\\ \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\tau} &\geq \mathbf{0} \end{aligned}$$

Let $(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\tau}})$ describe an optimal solution to $(D{:}SD_2)$. A (decomposed) *L-shaped optimality cut* is then defined as follows:

$$\Theta^k + \sum_{e \in E} (c_e^k - \tilde{\beta}_e) x_e^0 \geq \sum_{W \in \mathcal{W}^k} \Phi^k(W) \tilde{\alpha}_W - \sum_{e \in E} \tilde{\tau}_e \qquad \text{(LS)}$$

Notice that the right hand side of (LS) is a constant. Depending on its value, the obtained cut can be strengthened by rounding: the coefficients next to each x_e^0 can be replaced by $\min\{c_e^k - \tilde{\beta}_e, \sum_{W \in \mathcal{W}^k} \Phi^k(W)\tilde{\alpha}_W - \sum_{e \in E} \tilde{\tau}_e\}$, for each $e \in E$ and $k \in \mathcal{K}$.

4.2 Two-stage branch&cut algorithm

To describe the algorithm we use a slightly more general and compact notation: Let $\boldsymbol{x^0}$ and $\boldsymbol{x^k}$ be variable vectors for the first stage and scenario $k \in \mathcal{K}$, respectively. Moreover, let $\boldsymbol{c}^0$ be the objective coefficient vector in the first stage. With $\mathcal{X}_\nu^0$ being the first stage polyhedron in iteration ν defined by the separated L-shaped and integer optimality cuts let RMP denote the relaxed master problem, i.e.,

$$\min\{\boldsymbol{c}^0 \boldsymbol{x^0} + \Theta \mid \boldsymbol{x^0} \in \mathcal{X}_\nu^0, \Theta = \sum_{k \in \mathcal{K}} p^k \Theta^k, \boldsymbol{x^0} \geq \boldsymbol{0}, (\Theta^1, \dots, \Theta^K) \geq \boldsymbol{0}\}. \qquad \text{(RMP)}$$

Furthermore, let $(R)SP^k$ denote the (relaxed) subproblem, i.e., the restricted deterministic SNDP of scenario $k \in \mathcal{K}$. A brief description of the algorithm is given as follows:

Step 0: Initialization. $UB := +\infty$ (global upper bound, corresponding to a feasible solution), $\nu := 0$. Create the first pendant node. In the initial RMP the set of (integer) L-shaped cuts is empty.

Step 1: Selection. Select a pendant node from the branch&bound tree, if such a node exists. Otherwise STOP.

Step 2: Separation. Solve the RMP at the current node. $\nu := \nu+1$. Let $(\tilde{\boldsymbol{x}}_{\boldsymbol{\nu}}^{\boldsymbol{0}}, \tilde{\Theta}_\nu, \tilde{\Theta}_\nu^1, \dots, \tilde{\Theta}_\nu^K)$ be the current optimal solution.

(2.1) If $\boldsymbol{c}^0 \tilde{\boldsymbol{x}}_{\boldsymbol{\nu}}^{\boldsymbol{0}} + \tilde{\Theta}_\nu > UB$ fathom the current node and goto Step 1.

(2.2) Search for **violated L-shaped cuts**:
For all $k \in \mathcal{K}$, compute the LP-relaxation value $R(\tilde{\boldsymbol{x}}_{\boldsymbol{\nu}}^{\boldsymbol{0}}, k)$ of RSP^k. If $R(\tilde{\boldsymbol{x}}_{\boldsymbol{\nu}}^{\boldsymbol{0}}, k) > \tilde{\Theta}_\nu^k$: insert the rounded L-shaped cut (LS) into RMP.
If at least one L-shaped cut was inserted goto Step 2.

(2.3) If $\tilde{\boldsymbol{x}}_{\boldsymbol{\nu}}^{\boldsymbol{0}}$ is binary, search for **violated integer optimality cuts**:

(2.3.1) For all $k \in \mathcal{K}$ s.t. $\tilde{\boldsymbol{x}}_{\boldsymbol{\nu}}^{\boldsymbol{k}}$ was not binary in the previously computed LP-relaxation, solve SP^k to integer optimality: Let $Q(\tilde{\boldsymbol{x}}_{\boldsymbol{\nu}}^{\boldsymbol{0}}, k)$ be the optimal solution value.
If $\sum_{k \in \mathcal{K}} p^k Q(\tilde{\boldsymbol{x}}_{\boldsymbol{\nu}}^{\boldsymbol{0}}, k) > \tilde{\Theta}_\nu$ insert integer optimality cut (int-LS) into RMP. Goto Step 2.

(2.3.2) $UB := \min(UB, \boldsymbol{c}^0 \tilde{\boldsymbol{x}}_{\boldsymbol{\nu}}^{\boldsymbol{0}} + \tilde{\Theta}_\nu)$. Fathom the current node. Goto Step 1.

Step 3: Branching. Using a branching criterion create two nodes and append them to the list of pendant nodes. Goto Step 1.

The two types of generated cuts are L-shaped optimality cuts (LS) and integer optimality cuts (int-LS), as described in the next paragraph. Notice that we do not need to add any *feasibility cuts* since we are dealing with a problem with *complete recourse*, i.e., every first stage solution is feasible and can be augmented—and oriented—to a feasible scenario solution.

Integer optimality cuts. Let $(\tilde{\boldsymbol{x}}^{\mathbf{0}}_{\boldsymbol{\nu}}, \tilde{\Theta}_\nu, \tilde{\Theta}^1_\nu, \ldots, \tilde{\Theta}^K_\nu)$ be a first stage solution with $\tilde{\boldsymbol{x}}^{\mathbf{0}}_{\boldsymbol{\nu}}$ being binary and with the second stage value $Q(\tilde{\boldsymbol{x}}^{\mathbf{0}}_{\boldsymbol{\nu}}) := \sum_{k\in\mathcal{K}} p^k Q(\tilde{\boldsymbol{x}}^{\mathbf{0}}_{\boldsymbol{\nu}}, k)$. Let $\mathcal{I}_\nu := \{e \in E \mid \tilde{x}^0_{\nu,e} = 1\}$ be the index set of the edge variables chosen in the first stage, and the constant L be a known lower bound of the recourse function—for the SSNDP a feasible value is $L = 0$.

To explicitly cut off the solution $(\tilde{\boldsymbol{x}}^{\mathbf{0}}_{\boldsymbol{\nu}}, \tilde{\Theta}_\nu, \tilde{\Theta}^1_\nu, \ldots, \tilde{\Theta}^K_\nu)$ the general integer optimality cuts of the L-shaped scheme [19] are inserted:

$$\Theta \geq (Q(\tilde{\boldsymbol{x}}^{\mathbf{0}}_{\boldsymbol{\nu}}) - L)\left(\sum_{e\in\mathcal{I}_\nu} x^0_e - \sum_{e\in E\setminus\mathcal{I}_\nu} x^0_e - |\mathcal{I}_\nu| + 1\right) + L. \qquad \text{(int-LS)}$$

These cuts are quite weak since they almost only cut off the current first stage solution. However, these cuts are necessary for closing the integrality gap, cf. [19] (recall that we are dealing with an NP-hard second-stage problem with binary variables).

A second (even simpler) type of optimality cuts looks like follows:

$$\sum_{e\in\mathcal{I}_\nu} x^0_e - \sum_{e\in E\setminus\mathcal{I}_\nu} x^0_e \leq |\mathcal{I}_\nu| - 1 \qquad \text{(i-LS)}$$

These cuts (coined *combinatorial Benders cuts* in [7]) are even weaker because they do not contain the explicit bound on the Θ variable. But since the coefficients of these cuts are all binary they are numerically more stable. In our experiments, these cuts turn out being very important for avoiding numerical problems and tailing off effects, cf. Section 6.

5 Dual Lifting: Deriving strengthened L-shaped cuts

Since the relaxed master problem mainly consists of L-shaped optimality cuts, the number of master iterations of the 2-stage b&c approach—and hence, the overall running time—is highly influenced by the strength of the generated L-shaped cuts. In this paper we propose a new and fast way of strengthening the generated L-shaped cuts. Most of the previously proposed strengthening approaches (cf. [8,22,25,28,33]), require solving an auxiliary LP in order to generate a stronger L-shaped cut. With our new *dual lifting* approach, this is not the case—the procedure is very efficient and we are able to find a strenghtened L-shaped cut in linear time (with respect to the number of variables). Instead of optimizing an additional problem, the L-shaped cuts for the formulation (SD_2) of the SSNDP can be strengthened as follows: Notice that if for an edge $e \in E$ the current first stage solution satisfies $\tilde{x}^0_e = 0$, then the corresponding dual variable β_e does not appear in the objective function of the dual $(D{:}SD_2)$. Furthermore, the variables γ_e do not appear in the objective function neither. Hence, it is not difficult to see that we deal with a highly degenerated LP and one can expect that the optimal solutions to the dual subproblem $(D{:}SD_2)$ usually produce positive slacks in the constraints (D:2) (typically, if possible, dual variables with zero coefficients in the objective function will be fixed to zero by an LP solver). The idea is now to produce another LP optimal solution of the dual subproblem such that the corresponding slacks are reduced to zero. Therefore, the values of the dual multipliers (β_e) in the associated L-shaped cut will be increased as follows:

Let $(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\tau}})$ be an optimal solution to $(D{:}SD_2)$ as before. For all edges $e = \{i, j\} \in E$ set

$$\hat{\beta}_e := \begin{cases} \tilde{\gamma}_e - \max\limits_{a \in \{(i,j),(j,i)\}} \left\{ \sum\limits_{W \in \mathcal{W}^k : a \in \delta^-(W)} \tilde{\alpha}_W \right\} & \text{if } \tilde{x}^0_e = 0 \\ \tilde{\beta}_e & \text{otherwise.} \end{cases}$$

If $\hat{\beta}_e > \tilde{\beta}_e$ holds for at least one edge $e \in E$ the *lifted L-shaped cut* is given as:

$$\Theta^k + \sum_{e \in E} (c^k_e - \hat{\beta}_e) x^0_e \geq \sum_{W \in \mathcal{W}^k} \Phi^k(W) \tilde{\alpha}_W - \sum_{e \in E} \tilde{\tau}_e. \qquad \text{(l-LS)}$$

Theorem 6 *The lifted L-shaped cuts* (l-LS) *are valid and strictly stronger than the standard L-shaped cuts* (LS).

Proof Consider two L-shaped cuts: the standard one implied by the dual solution $(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\tau}})$ and the strengthened one $(\tilde{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\tau}})$ with $\hat{\boldsymbol{\beta}}$ being set as described above. Obviously, $(\tilde{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\tau}})$ is a feasible (and LP-optimal) solution to the dual subproblem $(D{:}SD_2)$ since $\hat{\boldsymbol{\beta}}$ is set without violating any dual constraints.

Furthermore, notice that $\hat{\beta}_e \geq \tilde{\beta}_e$, for all $e \in E$, and that the right-hand-side of both cuts is identical. Since there exists $e_1 \in E$ such that $\hat{\beta}_{e_1} > \tilde{\beta}_{e_1}$, the coefficient of $x^0_{e_1}$ is strictly smaller for the strengthened L-shaped cut than for the standard one which concludes the proof. □

It is well known that there is a trade-off between the invested running time for finding a Pareto-optimal L-shaped cut and its strength. One can easily construct an example where the lifted L-shaped cuts are not Pareto-optimal, i.e., they can be dominated by other L-shaped cuts with the same LP-value for $\tilde{\boldsymbol{x}}^{\mathbf{0}}$. Nonetheless, as it is demonstrated in the next section, our dual lifting procedure is a good heuristic alternative for strengthening L-shaped cuts without sacrificing the overall running time.

6 Computational results

Benchmark instances. To evaluate the performance of the 2-stage b&c in practice we focus on the restricted version of the SSNDP where connectivity requirements in each scenario $k \in \mathcal{K}$ are defined by nodes of type two (subset $\mathcal{R}^k_2 \subseteq V$), type one (subset $\mathcal{R}^k_1 \subseteq V$) and type zero ($V \setminus (\mathcal{R}^k_2 \cup \mathcal{R}^k_1)$). The main motivation for this choice is the application in the design of telecommunication networks where nodes of type two are important infrastructure nodes, or business customers, nodes of type one are single households, and nodes of type zero are, e.g., street intersections. For two distinct nodes u and v and each scenario $k \in \mathcal{K}$, the connectivity requirements are therefore: $r_{uv}^k = 2$ if both u and v are in $\mathcal{R}_2^k$, $r_{uv}^k = 1$ if one of them is in $\mathcal{R}_1^k$ and the other in $\mathcal{R}_1^k \cup \mathcal{R}_2^k$, and $r_{uv}^k = 0$, otherwise.

Deterministic instances were generated by adopting the idea of Johnson, Minkoff, and Phillips [16], which is frequently used as benchmark in the network design community. After randomly distributing $n \in \{30, 50, 75\}$ points in the unit square, a minimum spanning tree is computed using the points as nodes and the Euclidean distances between all vertex pairs as edge costs. To generate only feasible instances we augmented this tree by inserting edges between leaves which are adjacent in the planar embedding. The resulting biconnected graph is extended by adding all edges for which the Euclidean distance is less than or equal to $1.6\alpha/\sqrt{n}$. We have introduced α in order to control the density of the graph[1]. In our experiments we use $\alpha = 0.9$ which led to graphs with average density 2.07. The edge-connectivity requirements are set as follows. We have randomly drawn $\rho\%$ of the nodes as base sets of $\mathcal{R}_1$ and $\mathcal{R}_2$ customers. Here, we use $\rho = 40$ and we additionally introduce a random root node that is contained in $\mathcal{R}_2$. An example is given in Figure 2.

To transform these instances into stochastic ones we randomly and independently generate $\bar{k}$ scenarios. The probabilities are set by randomly distributing $10,000$ points over the scenarios, where each point corresponds to a probability of 0.01%. Edge costs $\boldsymbol{c^0}$ in the first stage are Euclidean distances and in the second stage for each edge e and scenario $k \in \mathcal{K}$ randomly drawn from $[1.1c_e^0, 1.3c_e^0]$. Edge-connectivity requirements are generated by randomly drawing $\rho^k\%$ from the vertex sets $\mathcal{R}_1$ and $\mathcal{R}_2$ each as $\mathcal{R}_1^k$ and $\mathcal{R}_2^k$ customers, respectively, for scenario k. Here, we use $\rho^k = 30$ for all scenarios k. The special root node was set to be an $\mathcal{R}_2^k$ node in each scenario k.

For each deterministic instance we generated a stochastic instance with $\bar{k} = 1000$ scenarios and took the first k to obtain an SSNDP instance with k scenarios[2]. Probabilities for the scenarios of the instances with $k < 1000$ are scaled appropriately. Overall, we generated 20 graphs for each $n \in \{30, 50, 75\}$ and k leading to 840 instances[3]. Due to the high computational effort we used 580 instances in the experiments: for $n = 30$ all 280 instances are used, for $n = 50$ instances with at most 250 scenarios (180 instances), and for $n = 75$ we used instances with at most 100 scenarios (120 instances).

Computational settings. We implemented the (single-stage) b&c and the 2-stage b&c for the strongest of the three presented models, namely (SD_2), considering the following settings:

– `EF`: (single-stage) direct b&c applied to the extended formulation without decomposition. Flow-based separation procedures known for the deterministic SNDP, cf., e.g., [31], are applied over all $k \in \mathcal{K}$.

[1] The original parameter used by [16] was 1.6 and corresponds to $\alpha = 1$ in our setting

[2] We use 14 values for k: $k \in \{5, 10, 20, 50, 75, 100, 150, 200, 250, 300, 400, 500, 750, 1000\}$.

[3] These instances can be downloaded from our SSNDP webpage, see [30]

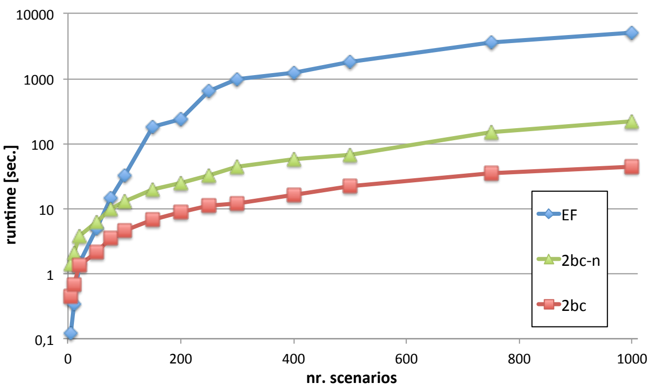


Fig. 5 Average running time for graphs with 30 nodes grouped by the number of scenarios. EF: direct approach (extended formulation), 2bc-n: 2-stage branch&cut without dual lifting, 2bc: 2-stage branch&cut with dual lifting.

– `2bc`: 2-stage b&c with the separation of lifted L-shaped cuts and integer L-shaped cuts. Each subproblem is solved with the flow-based separation procedures known for the deterministic SNDP.
– `2bc-n`: the same as 2-stage b&c, but without lifting the L-shaped cuts.

We used Abacus 3.0 as a generic branch&cut framework with IBM CPLEX (version 12.1) as LP solver via the interface COIN-Osi 0.102. All experiments were performed on an Intel Xeon 2.5 GHz machine with six cores and 64 GB RAM under Ubuntu 12.04. Each run was performed on a single core and the timelimit was set to 2 hours. All reported values are averages over 5 independent runs. For `2bc` and `2bc-n`, integer optimality cuts (i-LS) were included by default. Although theoretically weaker, these cuts are numerically more stable and turned out to be necessary in practice, in order to avoid numerical difficulties with some of the instances.

Computational benefits of the decomposition approach. Figures 5 and 6 show the comparison of the running times of the three considered settings. The average running times of `EF`, `2bc` and `2bc-n` for the instances with 30 nodes are given in Figure 5. We observe that for up to 50 scenarios, `EF` is faster, but for instances with 100 or more scenarios, both decomposition approaches clearly outperform `EF`. For example, on the instances with 1000 scenarios, `2bc` is on average about 22 times faster than `EF`. These results are consistent with the ones reported for the stochastic Steiner tree problem in [3]: For a lower number of scenarios, `EF` is superior due to the set-up overhead needed for the decomposition. With an increasing number of scenarios, the 2-stage b&c pays off and significantly outperforms the `EF` approach.

A similar behavior can also be observed on the sets of larger instances: the box plots in Figure 6 show the distribution of the running times for the instances with 50 and 75 nodes. For a fixed number of scenarios, the bottom and top of each box represent the first and third quartiles of the corresponding running times of 20 instances per group. Median running times are indicated by an horizontal line. We observe

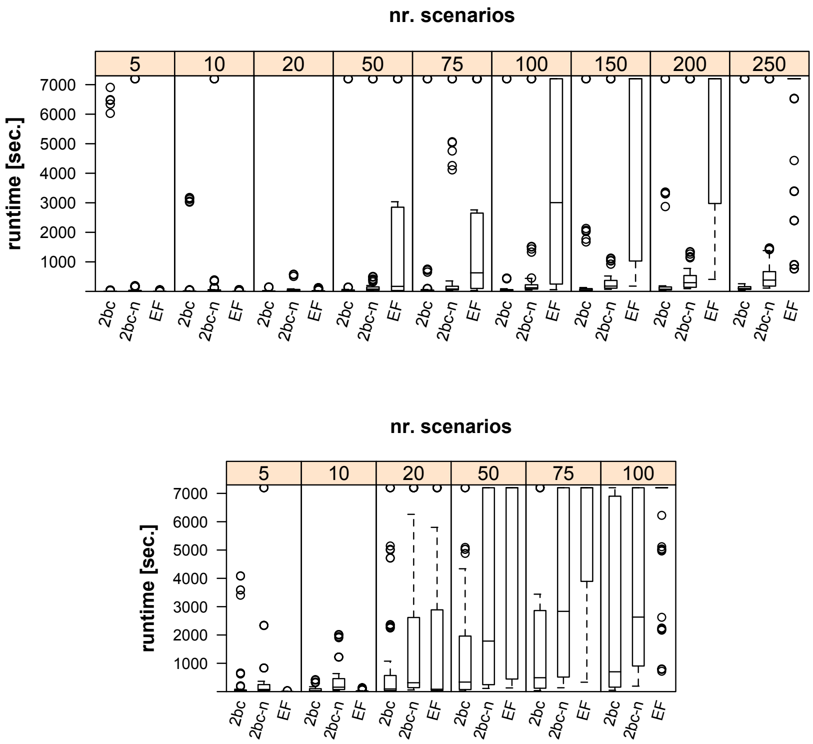


Fig. 6 Boxplots showing the running time for graphs with 50 nodes (top) and 75 nodes (below), respectively, grouped by number of scenarios.

that the number of scenarios from which on 2bc outperforms EF decreases with an increasing size of the input graphs.

While the performance of 2bc remains relatively stable, high dispersion and skewness of the running times for EF can be observed. More precisely, out of 20 instances with 50 nodes and 250 scenarios, 18 of them are solved after the timelimit of 2 hours using 2bc (the average running time of 2bc is 13 min.), whereas 15 of them remain unsolved with the EF approach (with an average running time of 101 min.). Similarly, for instances with 75 nodes and 100 scenarios, 2bc solves 15 out of 20 to optimality, and within the same time limit EF solves only 3. Outlier points of the decomposition are due to numerical issues; for all of these instances the optimum solution is known early but needs to be verified and many b&b nodes are generated.

Dual lifting vs. Pareto-optimal approach. Figures 5–6 also highlight the benefits of the dual lifting procedure. The running times of 2bc are always faster and less dispersed when compared with the running times of 2bc-n. Over all instances, the average speedup obtained trough dual lifting is about four, and the most significant speedup is about 16 when solving graphs with 75 nodes—again, the benefit increases with an increasing graph size.

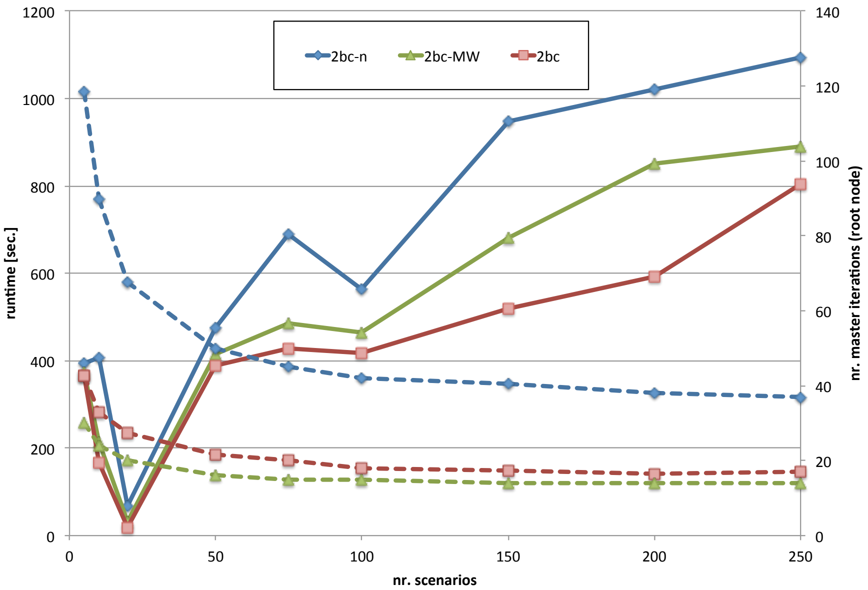


Fig. 7 Average running time (solid lines) and median number of master iterations (dashed lines) of the 2-stage b&c with cut strengthening (2bc, red lines), without cut strengthening (2bc-n, blue lines), and with the Magnanti-Wong method (2bc-MW, green lines) for graphs with 50 nodes.

A well-known and frequently used approach for strengthening L-shaped cuts is the method for finding Pareto-optimal cuts by Magnanti and Wong [22]. Figure 7 shows the average running times (solid lines) and the average number of master iterations (dashed lines) for graphs with 50 nodes and three approaches: `2bc-MW` (this is 2-stage b&c with Pareto-optimal L-shaped cuts added at each iteration), `2bc` and `2bc-n`. Comparing the number of master iterations, we note that the dual lifting reduces the number of master iterations of `2bc-n` by an average factor of 9 (for instances with 50 nodes). The number of master iterations is even better for `2bc-MW`, however, due to the overhead of solving additional LPs for finding Pareto-optimal cuts, the overall running time of `2bc-MW` is worse compared to the running time of `2bc`. We also notice that `2bc-MW` outperforms `2bc-n` in terms of the running time, which underlines the importance of the strengthening procedures in the generation of L-shaped cuts. Finally, we also tried to hybridize `2bc-MW` with `2bc`, but it turned out that this method does not improve the running time of `2bc`.

More detailed results regarding our computational experiments (including optimal solution values, running times, etc.) can be found at [30].

7 Discussion, conclusions and future work

In this paper we introduced the two-stage stochastic version of the survivable network design problem with a finite number of discrete scenarios and complete recourse. We presented three ways to model this problem: an undirected and two semi-directed formulations. For the undirected model we showed that facet defining inequalities of the deterministic counterparts can be lifted and yield facet defining inequalities of the stochastic model. This is the first result concerning the polyhedra of stochastic network design problems. The semi-directed formulations rely on orientation properties of edge-κ-connected graphs, and we prove that they are strictly stronger than the undirected formulation. We also discussed a 2-stage branch&cut algorithm using a decomposition approach. Moreover, a dual lifting procedure is introduced, representing an easy way for strengthening the L-shaped optimality cuts. Our computational study showed the benefits of using the decomposition for stochastic network design problems. Furthermore, the dual lifting technique highly reduces the number of master iterations and the computational running time. Compared to the generation of Pareto-optimal L-shaped cuts with the Magnanti-Wong method, our new dual lifting procedure requires slightly more master iterations, but outperforms the Magnanti-Wong method with respect to running time (due to the overhead of solving additional LPs for the latter approach).

Our computational experiments show that our two-stage branch-and-cut approach is more than an order of magnitude faster than the direct single-stage b&c approach. For the SSNDP, we are able to solve instances with up to 75 nodes, or 1000 scenarios, respectively, to provable optimality. We are convinced that our simple and fast dual lifting technique will be useful for general network optimization problems in the two-stage (stochastic optimization) setting.

An open problem remains how to generate Pareto-optimal L-shaped cuts by a combinatorial algorithm. Our dual lifting does not ensure the Pareto optimality—counter examples can be derived easily.

We remark that similar results can be obtained for the stochastic version of the node-connectivity $\{0, 1, 2\}$-SNDP, cf. Section 2. Only recently, in [5], new graph orientation properties have been given that allow to derive stronger directed MIP formulations for the deterministic $\{0, 1, 2\}$-SNDP with node-connectivity requirements. Using this result and analogously to formulation (SD_2), it is possible to formulate a stronger semi-directed model for the stochastic counterpart. L-shaped cuts and dual lifting can be derived from this model in a similar way, to be used within the 2-stage b&c.

Last but not least, we like to mention that several interesting problems are still open for the deterministic SNDP. For example, stronger directed formulations for higher node-connectivity requirements, i.e., > 2, or orienting solutions where both node- and edge-connectivity is required are still open problems. However, by using the same techniques presented in this article, any improvements in the deterministic context should be transferable to the related stochastic problems, formulations, and algorithms.

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