Hop Constrained Connected Facility Location: A Hierarchy of Formulations

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Abstract

Connected Facility Location (ConFL) is a problem that combines network design and facility location aspects: given a set of customers, a set of potential facility locations and some inter-connection nodes, ConFL searches for the minimum-cost way of assigning each customer to exactly one open facility, and connecting the open facilities via a Steiner tree. The costs needed for building the Steiner tree, facility opening costs and the assignment costs need to be minimized.

In the Hop Constrained Facility Location Problem (HC ConFL) the number of edges between the root and any open facility should not exceed a given number $H > 1$. We develop 16 mixed integer programming models for this problem. In branch-and-bound frameworks the quality of linear programming lower bounds of these formulations is of particular interest. In this theoretical study we compare the relative quality of these relaxations and provide a hierarchy of the corresponding models.

This paper comprises a first theoretical study on polyhedral aspects of this problem of great practical importance in the design of telecommunication or data-management networks.

Keywords:
Hop constrained Minimum Spanning trees, Hop constrained Steiner trees, Connected Facility Location, Mixed Integer Programming Models, LP-relaxations
90B18, 90C57, 90C10

1. Introduction

The Connected Facility Location problem (ConFL) is defined as follows: We are given an undirected graph (V, E) with $R \subset V$ being the set of *customers*, $F \subset V$ the set of *facilities*, and $S' = V \setminus (F \cup R)$ the set of intermediate nodes and the *root* node $r \in F$. There are two sets of edges: *core edges* $E_S \subseteq V \times V$ that can be used to connect the set of open facilities to the root, and *assignment edges* $E_R \subseteq F \times R$ that can be used to assign customers to open facilities. We are also given costs of core edges $c_e^c \geq 0$, $e \in E_S$, assignment costs $c_e^a \geq 0$, $e \in E_R$ and facility opening costs $f_j \geq 0$, $j \in F$. The root node is always considered as an open facility. The goal is to find a subset of open facilities such that:

- each customer is assigned to an open facility,

- a Steiner tree (consisting of core edges) connects all open facilities, and

- the sum of assignment, facility opening and Steiner tree costs is minimized.

If a facility node $j \in F$ is part of the core network without serving any customer, then j does not incur any opening costs and it is considered as an intermediate node (also referred to as *Steiner node*). Customer nodes may be used as Steiner nodes as well. In [1] we have shown that without loss of generality, one may assume that sets F, R and S' form a non-trivial partition of V. This implies that the sets of core and assignment edges will be disjoint ($E_R \cup E_S = E$, $E_R \cap E_S = \emptyset$), and hence we will use the same notation c_e, for $e \in E_S \cup E_R$, instead of c_e^c and c_e^a. By $S = F \cup S'$ we will denote the set of nodes building the *core network*.

If connection costs are non-negative, there always exists an optimal ConFL solution that obeys a tree structure. In such simply connected graphs, reliability against a single edge/node failure is not provided. More precisely, the probability that a communication will be interrupted by a link/node failure increases with the number of links/nodes in the path between the root and an installed facility. Typically, in applications of ConFL like content distribution networks [2] or telecommunication networks [1], economic arguments do not allow the installation of survivable networks with higher edge/node connectivity. Since paths with fewer hops have a better performance, these reliability constraints are modeled using *hop constraints*. In the tree representing a feasible ConFL solution, the number of edges on the path between the root node and an open facility is usually called the number of *hops*.

Based on this definition the *Hop Constrained Connected Facility Location Problem* (HC ConFL) is: Given an instance of the rooted ConFL and an integer number $H > 1$, find a minimum-cost solution that is valid for ConFL and in which there are at most H hops between the root and any open facility.

An instance of HC ConFL is shown in Figure 1a). Figure 1b) illustrates a feasible solution for $H \geq 2$. In this and all succeeding examples we use the following symbols: $\boxed{r}$ represents the root node, $\bigcirc$ represents a Steiner node, $\boxed{j}$ represents a facility j, $\diamond$ represents a customer. In these examples the default edge/arc values, facility opening and assignment costs are all set to one. Costs different from one are displayed next to the respective arc / node. The core network is presented as undirected graph.
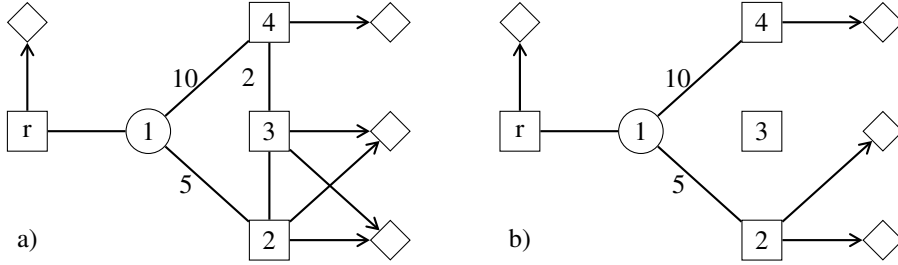

Figure 1: a) Original instance; b) Feasible solution.

1.1. Literature Review

The Hop Constrained Connected Facility Location Problem has first been considered in [3]. There we prove that the HC ConFL is not in APX and describe how to model the HC ConFL as Connected Facility Location problem on a layered graph. We give a polyhedral comparison of six cut set based and one compact formulation. In a computational study we show that the branch-and-cut approach, based on a weaker cut set model on a layered graph, computationally outperforms the layered graph model with stronger LP relaxation bounds.

Two combinatorial optimization problems closely related to HC ConFL, the *Connected Facility Location* problem and the *Steiner tree problem with hop constraints*, have been intensively studied in the literature.

The Connected Facility Location problem. Early work on ConFL mainly includes approximation algorithms. The currently best-known (constant) approximation ratio is provided by Eisenbrand et al. [4]. Ljubić [5] describes a hybrid heuristic combining Variable Neighborhood Search with a reactive tabu search method and compares it to an exact branch-and-cut approach and two new sets of test instances. Tomazic and Ljubić [6] present a Greedy Randomized Adaptive Search Procedure (GRASP) for the problem. Bardossy and Raghavan [7] develop a dual-based local search (DLS) heuristic for a generalization of the ConFL problem. Leitner and Raidl [8] present a branch-and-cut-and-price approach for a variant of ConFL with *capacities on facilities*.

In [1] we provide a complete hierarchy of ten MIP formulations with respect to the quality of their LP-bounds and compare them in a computational study. Formulations that model connectivity of customers give better lower bounds than their counterparts modeling connectivity of facilities but a cut set based model using the latter shows the best performance in practice.

The Steiner tree problem with hop constraints (HCSTP). In the hop constrained Steiner tree problem, the goal is to connect a given subset of customers at minimum cost, while using a subset of Steiner nodes, so that the number of hops between the root and each terminal does not exceed H. A large body of work has been done for the Minimum Spanning Tree problem with hop constraints (HCMST), a special case of the HCSTP where each node in the graph is a terminal. A recent survey for the HCMST can be found in [9]. Gouveia et al. [10] use a reformulation on layered graphs to develop the strongest MIP models known so far for the HCMST.

Much less has been said about the HCSTP: The earliest work is by Gouveia [11]. The author develops a strengthened multi-commodity flow model for HCMST and HCSTP. The obtained LP lower bounds are equal to those of a Lagrangean relaxation approach described in [12].

Voß [13] presents MIP formulations based on Miller-Tucker-Zemlin constraints. The models are strengthened by disaggregation. A simple heuristic and numerical results are also provided. Gouveia [14] gives a survey of hop-indexed tree and flow formulations for the hop constrained spanning and Steiner tree problem.

Costa et al. [15] compare a greedy heuristic, a destroy-and-repair algorithm and a tabu search approach for a HCSTP with revenues and budget constraints. In [16] the same authors introduce new formulations based on generalized subtour elimination constraints and a set of hop constraints of exponential size. They compare them to the models presented in Voß [13] and Gouveia [14] both theoretically and computationally.

1.2. Our Contribution

There are two ways to define hop constraints on a ConFL instance: a) there are at most H hops between the root and any open facility, or b) there are at most $H + 1$ hops between the root and any customer. Correspondingly, following one or the other concept, various formulations can be derived using, e.g., path or jump inequalities, multi-commodity flows or layered graphs to model hop constraints. We introduce 16 formulations to model HC ConFL and provide a full polyhedral comparison of these models. We thereby show that the approaches proposed in [3] are stronger than all other approaches found in the literature so far and that the models based on the customers-hops concept are in most cases stronger than their facility-hops based counterparts.

This paper comprises a first theoretical study on polyhedral aspects of this problem of great practical importance in the design of telecommunication or data-management networks.

2. (M)ILP Formulations for HC ConFL

Problem formulations on directed graphs often give better lower bounds than their undirected equivalents (see, e.g., [17]). By replacing edges between nodes in S by two directed arcs of the same cost and each edge between a facility and a customer by an arc directed from the facility towards the customer, undirected instances can be transformed into directed ones [1]. In the remainder of this paper we will focus on the Hop Constrained Connected Facility Location problem defined on a directed graph $G = (V, A)$ where $A = A_S \cup A_R$ and $A_R = \{jk \mid j \in F, k \in R, \{j, k\} \in E_R\}$, $A_S = \{ij, ji \mid e = \{i, j\} \in E_S\}$. We will refer to A_R as *assignment arcs* and to A_S as *core arcs*. Furthermore, for any $W \subset V$ we denote $\delta^-(W)$ and $\delta^+(W)$ by $\{ij \in A \mid i \notin W, j \in W\}$ and $\{ij \in A \mid i \in W, j \notin W\}$, respectively.

To model the problem, we will use the following binary variables:

$$x_{ij} = \begin{cases} 1, & \text{if } ij \text{ belongs to the solution} \\ 0, & \text{otherwise} \end{cases} \quad \forall ij \in A \qquad z_j = \begin{cases} 1, & \text{if } j \text{ is open} \\ 0, & \text{otherwise} \end{cases} \quad \forall j \in F$$

We define $x(D) = \sum_{ij \in D} x_{ij}$, for every $D \subseteq A$. By $\mathcal{P}(.)$ we denote the polytope of the LP-relaxation of any of the MIP models described in the following. By $\mathcal{P}_{x,z}(.)$ we denote the orthogonal projection of that polytope onto the space of variables $\mathbf{x}$ and $\mathbf{z}$. By $\upsilon_{LP}(.)$ we denote the value of the optimal LP-solution over the polytope $\mathcal{P}(.)$. We call formulation A stronger than formulation B if $\upsilon_{LP}(A) \geq \upsilon_{LP}(B)$ for all instances of HC ConFL. The relation is referred to as strictly stronger, if there exist instances for which $\upsilon_{LP}(A) > \upsilon_{LP}(B)$. A constraint set C is referred to as (strictly) dominating constraint set D if the model obtained by replacing D by C is (strictly) stronger than the original one.

2.1. Cut Set Based Formulations for ConFL

To model the ConFL problem *without* hop constraints, we may follow two concepts to describe connectivity: (a) *Customer based* approach: we ensure that there is a path between the root and each customer, where in addition, facilities adjacent to customers in the solution need to be open, or (b) *Facility based* approach: we ensure that the solution defined on the assignment graph A_R is a feasible facility location solution, and in addition, there is a path between the root and each open facility.

Considering corresponding ConFL models proposed in the literature, the best lower bounds can be obtained using

the model based on directed cut set inequalities for the customer based approach, which is given as follows:

$$(CUT_R) \quad \min f(x, z) = \sum_{ij \in A} c_{ij} x_{ij} + \sum_{j \in F} f_j z_j$$

$$\text{s.t.} \quad \sum_{uv \in \delta^-(W)} x_{uv} + \sum_{jk \in A_R : j \notin W} x_{jk} \geq 1 \qquad \forall W \subseteq S \setminus \{r\}, W \cap F \neq \emptyset, \; \forall k \in R \tag{CR}$$

$$\sum_{jk \in A_R} x_{jk} = 1 \qquad \forall k \in R \tag{1a}$$

$$x_{jk} \leq z_j \qquad \forall jk \in A_R \tag{1b}$$

$$z_r = 1 \tag{1c}$$

$$x_{ij} \in \{0, 1\} \quad \forall ij \in A_S \tag{1d}$$

$$x_{jk} \in [0, 1] \quad \forall jk \in A_R \tag{1e}$$

$$z_j \in \{0, 1\} \quad \forall j \in F \tag{1f}$$

The objective comprises the cost for the Steiner arborescence (i.e., the *cost of the core network*, $\sum_{ij \in A_S} c_{ij} x_{ij}$), the cost to connect customers to facilities (that we also refer to as *assignment cost*, i.e. $\sum_{ij \in A_R} c_{ij} x_{ij}$) and the *facility opening cost* ($\sum_{j \in F} f_j z_j$). Inequalities (CR) represent the set of connectivity cuts. For every subset $W \subseteq S \setminus \{r\}$ and for each customer $k \in R$, an open arc from a facility in W toward j, necessitates a directed path from r towards W. Constraints (1a) ensure that every customer is connected to one facility, constraints (1b) guarantee that each facility is opened if a customer is assigned to it, equation (1c) defines the root node. Constraints (1a) are redundant in case that $c_{jk} > 0$ for all $jk \in A_R$.

If we replace (CR) by the following constraints,

$$\sum_{uv \in \delta^-(W)} x_{uv} \geq z_j \qquad \forall W \subseteq S \setminus \{r\}, \; \forall j \in W \cap F \neq \emptyset \tag{CF}$$

we obtain a facility based cut set model. We refer to it as CUT_F. We have shown in [1] that the LP-relaxation lower bounds of this formulation may be $|F| - 1$ times worse than the bounds of CUT_R. HC ConFL contains ConFL as a special case. Thus, this result still holds for HC ConFL as well.

When considering models involving variables **x** and **z** only, there are two ways to extend the cut set based formulations to model hop constraints: (i) path based and (ii) jump based inequalities. The earlier have been mentioned by Costa et al. [16], the latter are a development of Dahl et al. [9]. Both classes of inequalities are of exponential size. They can be used to model facility based or customer based hop constraints. We discuss these variants below.

2.1.1. Cut Set Formulations with Path Constraints

Let $P = \{(i_1, j_1), \ldots, (i_l, j_l)\}$ with $i_1 = r$, $j_l \in F \setminus \{r\}$ and $j_{k-1} = i_k$, $k = 2 \ldots l$ denote a simple path with l arcs between the root node and a facility. For a given number l, let $\mathcal{P}_l$ be the set of all such paths P consisting of l arcs.

Facility based path constraints. Observe that for any path on the core graph consisting of $H + 1$ arcs at most H arcs are allowed to be open in a valid solution. This can be ensured using the following *facility based path constraints*:

$$\sum_{uv \in P} (x_{uv} + x_{vu}) \leq H \quad \forall P \in \mathcal{P}_{H+1}, P \subseteq A_S \tag{P1}$$

These constraints are a lifted version of the path constraints

$$\sum_{uv \in P} x_{uv} \leq H \quad \forall P \in \mathcal{P}_{H+1}, \; P \subseteq A_S \tag{P2}$$

that have been proposed in [16] for a related variant of the hop constrained Steiner tree problem. Our new path constraints (P1) are even strictly stronger than (P2) as is shown in the example in Figure 2.

Customer Based Path constraints. When considering the graph G, only rooted paths whose length does not exceed $H + 1$ are allowed. Therefore, we can ensure that the number of hops between the root and any customer does not exceed $H + 1$ using the following *customer based path constraints*:

$$\sum_{uv \in P} (x_{uv} + x_{vu}) + x_{jk} \leq H + 1 \quad \forall P \in \mathcal{P}_{H+1}, \ P \subseteq A_S, \ jk \in A_R \tag{P3}$$

For design variables **x** taking values less or equal than 1, constraints (P3) are implied by (P1). Hence, in the remainder of this paper we will only consider facility based path constraints.
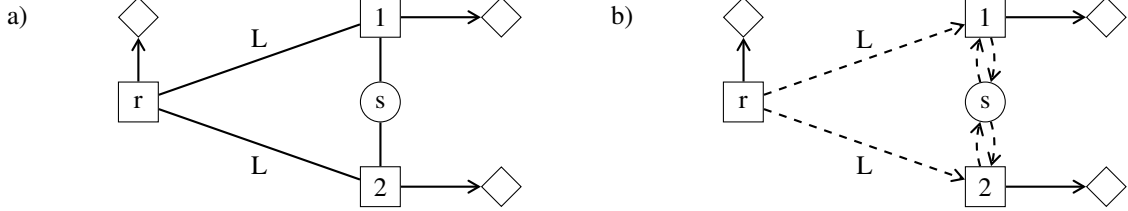

Figure 2: a) An instance for HC ConFL for which the hop limit H is 2. b) A solution for the instance in a) in which $x_{r1} = x_{r2} = x_{1s} = x_{s2} = x_{2s} = x_{s1} = 0.5$. It is only valid for model CUT_R (or CUT_F) in which constraints (P1) are replaced by the weaker constraints (P2).

The cut set formulation CUT_R (CUT_F) in which hop constraints are modeled using facility based path constraints will be denoted by CUT_R^P (CUT_F^P).

Lemma 1. $v_{LP}(CUT_R^P) \geq v_{LP}(CUT_F^P)$.

Proof. Any LP-optimal solution of the model CUT_R is also feasible for the model CUT_F and has the same objective value. Since the constraints (P1) are common in both models, the result follows immediately. $\square$

2.1.2. Cut Set Formulations with Jump Constraints

To formulate cut set based models for HC ConFL with jump constraints we use the notation proposed in [9]. Again, we will propose two classes of jump constraints derived on the ideas of measuring the distance between the root and (i) an open facility, or (ii) a customer.

Facility based jump constraints. Let $S_0, S_1, \ldots, S_{H+1}$ be a partition of S into (possibly empty) subsets but such that $r \in S_0$ and $S_{H+1} \cap F \neq \emptyset$. We call $J = J(S_0, S_1, \ldots, S_{H+1}) = \bigcup_{(i,j):i<j-1}[S_i, S_j]$ where $[S_i, S_j] = \{uv \in A : u \in S_i, v \in S_j\}$ a H-jump. Using J_H, the set of all possible H-jumps, we can formulate hop constraints on the core graph by using the following jump inequalities:

$$\sum_{ij \in J} x_{ij} \geq z_l \quad \forall J \in J_H, \ S_{H+1} = \{l\}, l \in F. \tag{JF}$$

Given a feasible ConFL solution, constraints (JF) ensure that for any open facility, at least one edge in each of the jumps is used, i.e., the path between the root and the facility is at most H.

Lemma 2. *Every jump constraint defined on a partition of the nodes such that $|S_{H+1}| \geq 2$ and $S_{H+1} \cap F \neq \emptyset$ is implied by a jump constraint for a partition of the nodes such that $S_{H+1} = \{l\}, l \in F$.*

Proof. The set of arcs in the jump can only become smaller when a node from S_{H+1} is moved to S_H. $\square$

Lemma 3. *Connectivity cuts* (CF) *are contained in the family of facility based jump constraints if empty subsets are allowed in the partitions defining the jumps.*

Proof. An inequality (CF) for $W \subseteq S \setminus \{r\}$ is the jump inequality for the following partition of the node set: $S_{H+1} = \{j\}$ such that $j \in W \cap F$, $S_H = W \setminus \{j\}$, $S_0 = S \setminus W$ and $S_i = \emptyset$ for $i = 1, \ldots, H - 1$. $\square$

To the best of our knowledge, this class of jump constraints involving arc and node variables has not been used in the literature so far. Besides HC ConFL, these inequalities are of particular importance when modeling other hop constrained network design problems with node variables, like the hop constrained prize-collecting STP or STP with revenues, budget and hop constraints (see, e.g., [15]). In the following, let CUT_F^J denote the formulation given by replacing constraints (P1) by (JF) in formulation CUT_F^P.

An illustration of the jump set for $J = J(S_0, S_1, \ldots, S_4)$ is given in Figure 3.
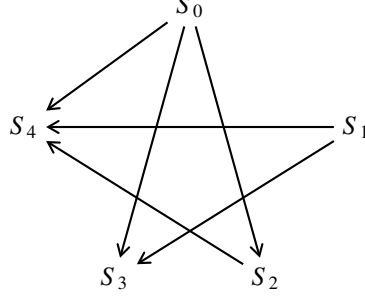
Figure 3: Illustration of the arcs contained in a jump for $H = 3$.

Customer based jump constraints. Let $S_0, S_1, \ldots, S_{H+2}$ be a partition of S such that $S_{H+2} = \{k\}$, $k \in R$ and $\{r\} \cup R \setminus \{k\} \subseteq S_0$. We call $J = J(S_0, S_1, \ldots, S_{H+2}) = \bigcup_{(i,j):i<j-1}[S_i, S_j]$ a $(H+1)$-jump. Using J_{H+1}, the set of all possible $(H+1)$-jumps, we can formulate hop constraints on the core and assignment graph by using the following jump inequalities.

$$\sum_{ij \in J} x_{ij} \geq 1 \quad \forall J \in J_{H+1}. \tag{JR}$$

Lemma 4. *Connectivity cuts* (CR) *are contained in the family of customer based jump constraints if empty subsets are allowed in the partitions defining the jumps.*

Proof. An inequality (CR) for $W \subseteq S \setminus \{r\}$ and $k \in R$ is the jump inequality for the following partition of the node set: $S_{H+2} = \{k\}$, $S_{H+1} = W$, $S_0 = V \setminus (W \cup \{k\})$ and $S_i = \emptyset$ for $i = 1, \ldots, H - 1$. $\square$

Lemma 5. $v_{LP}(CUT_R^J) \geq v_{LP}(CUT_F^J)$.

Proof. See the argument in the proof of Lemma 1. $\square$

Lemma 6. *Under LP optimality conditions* ($\forall j \in F : z_j > 0 \Rightarrow \exists k \in R : x_{jk} = z_j$), *customer based jump constraints* (JR) *are strictly stronger than the facility based ones,* (JF).

Proof. Let $(\mathbf{x}, \mathbf{z})$ be a vector satisfying customer based jumps and also such that $\forall j \in F : z_j > 0 \Rightarrow \exists k \in R : x_{jk} = z_j$. Then, we will show that $(\mathbf{x}, \mathbf{z})$ also satisfies facility based jump constraints, for any given H-jump partition set $J = J(S_0, S_1, \ldots, S_{H+1} = \{j\})$, $j \in F$, $r \in S_0$. Assume the opposite, i.e., that there exists a facility jump $J \in J_H$ violated by $(\mathbf{x}, \mathbf{z})$, which means $x(J) < z_j$. The customer jump $J' = J'(S_0, S_1, \ldots, S_{H+1} = \{j\}, S_{H+2} = \{k\})$ for k chosen such that $x_{jk} = z_j$ is then also violated:

$$x(J') = x(J) + \sum_{i \in F \setminus \{j\}} x_{ik} \stackrel{(1a)}{=} x(J) + 1 - x_{jk} < z_j + 1 - z_j = 1,$$

which is a contradiction. To show that the constraints (JR) are strictly stronger than (JF), consider the example in Figure 4 a): The customer jump constraint for the partition into subsets $S_0 = \{r\}$, $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3, 4\}$ and $S_4 = \{c\}$ is violated by the solution depicted in Figure 4 b). $\square$
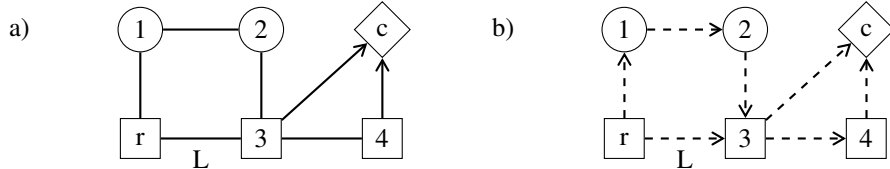

Figure 4: a) HC ConFL instance for which $H = 2$; b) Solution that is cut off by customer jumps, but not by facility jumps. Dashed lines correspond to $x_{ij} = 0.5$, $ij \in A$.

The formulation derived by replacing inequalities (P1) by (JR) in CUT_R^P will be denoted by CUT_R^J. It is an open question whether the separation of jump constraints is polynomially solvable.

2.2. Flow-based Formulations

In this section we present several ways to model HC ConFL using flow-based formulations.

2.2.1. Multi-Commodity Flow Formulations

Balakrishnan and Altinkemer [18] and Gouveia [12] have used multi-commodity flow formulations for network design problems with hop constraints. In both papers the authors limit the amount of flow for each commodity by the hop limit. Together with flow preservation constraints, this idea can be used to derive two valid MIP models for HC ConFL. In the facility based model, commodities will correspond to open facilities, in the customer based model, commodities will be customers.

Multi-Commodity Flow with One Commodity per Facility. Choosing one commodity per facility, each variable indicating an open facility is linked to a distinct commodity. A multi-commodity flow formulation with one commodity per each facility ensures connectivity of the solution and limits the number of hops at the same time. Variables g_{ij}^k will be set to one if facility $k \in F$ is open and a path between the root and k uses arc $ij \in A_S$. The model is given by:

$$(MCF_F) \quad \min \sum_{ij \in A} c_{ij} x_{ij} + \sum_{j \in F} f_j z_j$$

$$\text{s.t.} \quad \sum_{ji \in A_S} g_{ji}^k - \sum_{ij \in A_S} g_{ij}^k = \begin{cases} z_k & i = k \\ -z_k & i = r \\ 0 & i \neq k, r \end{cases} \quad \forall i \in S \quad \forall k \in F \setminus \{r\} \tag{2a}$$

$$0 \leq g_{ij}^k \leq x_{ij} \quad \forall ij \in A_S, \ \forall k \in F \setminus \{r\} \tag{2b}$$

$$\sum_{ij \in A_S} g_{ij}^k \leq H \quad \forall k \in F \setminus \{r\} \tag{2c}$$

$$(1a) - (1f)$$

Equations (2a) are the flow preservation constraints defining the flow from the root node to each facility. These constraints ensure the existence of a connected path from r to every open facility. The coupling constraints (2b) guarantee that the arc is open if a flow is sent through it. The maximum number of hops on the path from r to k is modeled by inequalities (2c). The remaining constraints ensure that the solution on the assignment subgraph induced by A_R is a feasible facility location solution.

Constraints (2c) can be replaced by stronger ones,

$$\sum_{ij \in A_S} g_{ij}^k \leq H \cdot z_k \quad \forall k \in F \setminus \{r\}. \tag{3}$$

Thereby we obtain a formulation that we denote by MCF_F^+.

Multi-Commodity Flow with One Commodity per Customer. Another choice for the commodities we use, is the set of customers. Assigning a commodity of demand 1 to each customer allows to remove the **z** variables from the flow preservation constraints. We obtain formulation MCF_R by replacing (2) by the following set of constraints:

$$\sum_{ji \in A} f_{ji}^k - \sum_{ij \in A} f_{ij}^k = \begin{cases} 1 & i = k \\ -1 & i = r \\ 0 & i \neq k, r \end{cases} \quad \forall i \in V \quad \forall k \in R \tag{4a}$$

$$0 \leq f_{ij}^k \leq x_{ij} \quad \forall ij \in A, \ \forall k \in R \tag{4b}$$

$$\sum_{ij \in A} f_{ij}^k \leq H + 1 \ \forall k \in R \tag{4c}$$

Constraints (4a) and (4b) guarantee the existence of a directed path from the root r to customer k. Together with constraints (4c) this path contains at most $H + 1$ arcs.

Note that in this formulation variables x_{jk} can be replaced by flows f_{jk}^k for all jk in A_R, as we have already shown in [1]. Also note that given the assignment (1d) and coupling constraints (4b) for $ij \in A_R$, inequalities (4c) are equivalent with the following inequalities in which flow is only restricted on the core graph:

$$\sum_{ij \in A_S} f_{ij}^k \leq H \quad \forall k \in R \tag{5}$$

Lemma 7. $\upsilon_{LP}(MCF_R) \geq \upsilon_{LP}(MCF_F)$. *Furthermore, there exist HC ConFL instances for which the strict inequality holds.*

Proof. We show that every LP-optimal solution $(\mathbf{x}, \mathbf{z}, \mathbf{f})$ in $\mathcal{P}(MCF_R)$ can be projected into a feasible solution from $\mathcal{P}(MCF_F)$ by decomposing the flows to the customers into facility flows. Given an LP-optimal solution $(\mathbf{x}, \mathbf{z}, \mathbf{f}) \in \mathcal{P}(MCF_R)$ we define the capacities on the subgraph $G_S = (S, A_S)$ as x_{ij}, for all $ij \in A_S$. Since $x_{ij} = max_{k \in R} f_{ij}^k$, and $z_j = max_{jk \in A_R} x_{jk}$, there will be enough capacity to independently route z_j units of flow, for all $j \in F$, such that $z_j > 0$. Now, we are going to construct $(\mathbf{x}, \mathbf{z}, \mathbf{g}) \in \mathcal{P}(MCF_F)$ as follows: We fix the ordering of the outgoing arcs of every node $i \in S$ and then apply an adapted Ford-Fulkerson maximum flow algorithm. To define $\mathbf{g}$, we send z_j units of flow from r towards $j \in F$, for all $j \in F$ such that $z_j > 0$. When searching for augmenting paths, we always follow the fixed ordering. Therefore, the outgoing arcs of a node always get saturated in the same order, independently of the commodity under consideration.

From equations (4a) we have $\sum_{jk \in A_R} f_{jk}^k = 1$ for all $k \in R$ and thus also $\sum_{ij \in A_S} f_{ij}^k \leq H$ holds for all $k \in R$. The objective value of the feasible solution $(\mathbf{x}, \mathbf{z}, \mathbf{g})$ is the same as $\upsilon_{LP}(MCF_R)$ and it is at most the value of the optimal LP-solution over $\mathcal{P}(MCF_F)$, which concludes this part of the proof.

Example 1 in Figure 6 shows a HC ConFL instance such that $\upsilon_{LP}(MCF_R) = 28 > 18 = \upsilon_{LP}(MCF_F)$. $\qquad\square$

2.2.2. Hop Indexed Multi-Commodity Flow Formulations

We now present two four-index models that are obtained from the previous ones by disaggregating flow variables for each arc and each commodity according to the arc's distance from the root. This idea has been originally proposed by Gouveia [11] where he developed a hop indexed formulation for the HCMST and HCSTP.

As for the MCF models, there are two choices on the commodities considered, facilities or customers. The variant in which facilities resemble commodities is a disaggregation of MCF_F, the other one is based on MCF_R.

Hop Indexed Multi-Commodity Flow Between Root and Facilities. Let g_{ij}^{kp} denote the flow towards facility $k \in F$, over arc ij, at position p on the path from r to k. Then formulation HD_F using hop-indexed multi-commodity flows from the root to facilities is given by replacing (2a)-(2c) with the following set of constraints:

$$\sum_{ji \in A_S} g_{ji}^{k,p-1} - \sum_{ij \in A_S} g_{ij}^{kp} = 0 \qquad \forall k \in F \setminus \{r\}, \ i \in S \setminus \{r,k\}, \ p \in \mathbf{I}_2^H \tag{6a}$$

$$-\sum_{rj \in A_S} g_{rj}^{k1} = -z_k \qquad \forall k \in F \setminus \{r\} \tag{6b}$$

$$\sum_{p=1}^{H} \sum_{jk \in A_S} g_{jk}^{kp} = z_k \qquad \forall k \in F \setminus \{r\} \tag{6c}$$

$$g_{ij}^{kp} = 0 \qquad \forall ij \in A_S, \ k \in F \setminus \{r\}, \begin{cases} i \neq r, \ p = 1 \\ i = r, \ p \in \mathbf{I}_2^H \end{cases} \tag{6d}$$

$$\sum_{p=1}^{H} g_{ij}^{kp} \leq x_{ij} \qquad \forall ij \in A_S, \ k \in F \setminus \{r\} \tag{6e}$$

$$g_{ij}^{kp} \geq 0 \qquad \forall ij \in A_S, \ k \in F \setminus \{r\}, \ p \in \mathbf{I}_1^H \tag{6f}$$

Equations (6a) - (6c) are flow conservation constraints. Equalities (6a) set the outflows of a commodity equal to the inflows of the same commodity one position earlier. Constraints (6b) ensure that z_k units of commodity k leave the root, constraints (6c) ensure they terminate in the respective facility. Constraints (6d) fix some flows to zero: Flows at position one are limited to arcs emanating from the root, flows at a higher position than one don't emanate from the root. Inequalities (6e) ensure an arc is in the solution if flow is sent through it.

In contrast to the model in [11] we do not consider variables g_{kk}^{kp} in our model. Thus, commodity flows can end in the respective facility at any position. All flows fixed to zero in (6d) could be removed from the model but they are kept to simplify the notation of constraints (6a)-(6c).

Hop Indexed Multi-Commodity Flow Between Root and Customers. Based on the MCF_R model, we can now derive a different hop-indexed formulation. Let f_{ij}^{kp} denote the flow towards customer $k \in R$, over arc ij, at position p of the path from r to k. Disaggregation HD_R of model MCF_R is obtained by replacing constraints (4a)-(4c) by

the following:

$$\sum_{ji \in A_S} f_{ji}^{k,p-1} - \sum_{ij \in A} f_{ij}^{kp} = 0 \qquad \forall i \in S \setminus \{r\},\ k \in R,\ p \in \mathbf{I}_2^{H+1} \tag{7a}$$

$$\sum_{rj \in A} f_{rj}^{k1} = 1 \qquad \forall k \in R \tag{7b}$$

$$\sum_{p=1}^{H+1} \sum_{jk \in A_R} f_{jk}^{kp} = 1 \qquad \forall k \in R \tag{7c}$$

$$f_{ij}^{kp} = 0 \qquad \forall ij \in A,\ k \in R, \begin{cases} i \neq r,\ p = 1 \\ i = r,\ p \in \mathbf{I}_2^{H+1} \\ ij \in A_S,\ p = H+1 \end{cases} \tag{7d}$$

$$\sum_{p=1}^{H+1} f_{ij}^{kp} \leq x_{ij} \qquad \forall ij \in A,\ k \in R \tag{7e}$$

$$f_{ij}^{kp} \geq 0 \qquad \forall ij \in A,\ k \in R,\ p \in \mathbf{I}_1^{H+1} \tag{7f}$$

Constraints (7a), (7b) and (7c) are flow preservation constraints similar to the ones in HD_F. Constraints (7d) fix some flows to zero as in HD_F: Flows at position one are only allowed to emanate from the root node. No flows in a later position can occur on arcs leaving the root. Inequalities (7e) ensure an arc is in the solution if there is flow on it.

Lemma 8. $v_{LP}(HD_R) \geq v_{LP}(HD_F)$. *Furthermore, there exist HC ConFL instances for which the strict inequality holds.*

Proof. Every LP-optimal solution $(\mathbf{x}, \mathbf{z}, \mathbf{f})$ in $\mathcal{P}(HD_R)$ can be projected into a feasible solution from $\mathcal{P}(HD_F)$ using a similar procedure as the one described in the proof of Lemma 7. Example 1 in Figure 6 shows a HC ConFL instance such that $v_{LP}(HD_R) = 28 > 18 = v_{LP}(HD_F)$. □

2.3. A Formulation Based on Subtour Elimination Constraints

Miller-Tucker-Zemlin Formulation. Miller-Tucker-Zemlin constraints [19] have been applied to a number of problems. Besides Connected Facility Location [1] we shall mention the models for the Hop Constrained Minimum Spanning and Steiner Tree Problem [16, 20]. In addition to variables $\mathbf{x}$ and $\mathbf{z}$, we now introduce *hop variables* $u_i \geq 0$, for all $i \in S$. These indicate the distance in hops of each node i from the root. The root node has a distance of zero.

Using the Miller-Tucker-Zemlin (*MTZ*) constraints (see, e.g., [14]), HC ConFL can be stated as $\min f(x, z)$ subject to

$$(MTZ) \qquad Hx_{ij} + u_i \leq u_j + (H-1) \qquad \forall ij \in A_S \tag{8a}$$

$$\sum_{ij \in A_S, i \neq k} x_{ij} \geq x_{jk} \qquad \forall j \in S \setminus \{r\},\ k \in V \tag{8b}$$

$$u_r = 0 \tag{8c}$$

$$1 \leq u_i \leq H \qquad \forall i \in S \setminus \{r\} \tag{8d}$$

$$(1a) - (1f)$$

Constraints (8a) are Miller-Tucker-Zemlin subtour elimination constraints, setting the difference $u_j - u_i$ for an open arc ij to at least 1. They thereby eliminate cycles in the Steiner tree connecting the facilities and paths on the core graph with more than H arcs. Constraints (8b) limit the out-degree of a node by its in-degree. Constraint (8c) sets the hop variable to zero for the root node.

It is not difficult to see that customer based MTZ formulation leads to the model with the same quality of lower bounds. We therefore consider only one MTZ model here.

2.4. Hop-Indexed Tree Formulations

By disaggregating arc variables in the MTZ formulation according to their distance from the root node, we obtain a model which is known as the *hop-indexed tree model* (see Gouveia [14] and Voß [13]). To model HC ConFL, there are two options for the hop-indexed variables. We can consider them on the whole graph or alternatively we can separate core and assignment graph and link them by the $\mathbf{z}$-variables indicating the use of facilities.

Hop-Indexed Tree on the Entire Graph. Let X_{ij}^p indicate whether arc $ij \in A$ is used at the p-th position from the root node. Then we can model HC ConFL by $\min f(x, z)$ subject to:

$$(HOP_R) \qquad \sum_{\substack{i \in S \setminus \{k\}: \\ ij \in A_S}} X_{ij}^{p-1} \geq X_{jk}^p \qquad \forall jk \in A,\ j \neq r,\ p \in \mathbf{I}_2^{H+1} \tag{9a}$$

$$\sum_{p=1}^{H+1} \sum_{jk \in A_R} X_{jk}^p = 1 \qquad \forall k \in R \tag{9b}$$

$$\sum_{p=1}^{H+1} X_{jk}^p \leq z_j \qquad \forall jk \in A_R,\ j \neq r \tag{9c}$$

$$\sum_{p=1}^{H+1} X_{ij}^p = x_{ij} \qquad \forall ij \in A \tag{9d}$$

$$X_{ij}^p = 0 \qquad \forall ij \in A, \begin{cases} ij \in A_S,\ p = H+1 \\ i = r,\ p \in \mathbf{I}_2^{H+1} \\ i \neq r,\ p = 1 \end{cases} \tag{9e}$$

$$X_{ij}^p \in \{0, 1\} \quad \forall ij \in A,\ p \in \mathbf{I}_1^{H+1} \tag{9f}$$

Constraints (9a) are connectivity constraints. As X_{ij}^p are binary, they eliminate cycles as well. Constraints (9c) ensure a facility is open if it serves a customer. Constraints (9b) ensure that each customer is served. Equations (9e) fix some of the X_{ij}^p to zero: Arcs emanating from the root can only be 1 hop away from it. Conversely, all other arcs are at least two hops away from the root. Equalities (9d) link the disaggregated variables X to the variables in the original space.

Hop-Indexed Trees on the Core Graph. As in the previous facility based approaches we separate core and assignment graph and link them by variables z_j, $j \in F$. After replacing variables X_{jk}^p, $jk \in A_R$ from formulation HOP_R by assignment variables x_{jk}, $jk \in A_R$, we can formulate HC ConFL using hop constraints only on the core graph as $\min f(x, z)$ such that

$$(HOP_F) \qquad \sum_{\substack{i \in S \setminus \{k\}: \\ ij \in A_S}} X_{ij}^{p-1} \geq X_{jk}^p \qquad \forall jk \in A_S,\ j \neq r,\ p \in \mathbf{I}_2^H \tag{10a}$$

$$\sum_{ij \in A_S} \sum_{p=1}^H X_{ij}^p \geq z_j \qquad \forall j \in F \setminus \{r\} \tag{10b}$$

$$\sum_{p=1}^H {X'}_{ij}^p = x_{ij} \qquad \forall ij \in A_S \tag{10c}$$

$$X_{ij}^p = 0 \qquad ij \in A_S, \begin{cases} i = r,\ p \in \mathbf{I}_2^H \\ i \neq r,\ p = 1 \end{cases} \tag{10d}$$

$$X_{ij}^p \in \{0, 1\} \quad \forall ij \in A_S,\ p \in \mathbf{I}_1^H \tag{10e}$$

$$(1a) - (1c), (1e), (1f)$$

Constraints (10a) are connectivity constraints like (9a). Similarly, inequalities (10b) link open facilities to their in-degree. Constraints (10d) are similar to (9e). Equalities (10c) link the disaggregate variables X to the variables in the original space.

Lemma 9. $\upsilon_{LP}(HOP_R) = \upsilon_{LP}(HOP_F)$.

Proof. To prove the relation we describe a mapping from any LP optimal solution of HOP_F to a solution of HOP_R of the same objective value and vice versa.

$\upsilon_{LP}(HOP_R) \geq \upsilon_{LP}(HOP_F)$: Let $(\mathbf{X}, \mathbf{z}) \in \mathcal{P}(HOP_R)$ and LP optimal. Then for any facility $j \in F$ equality $z_j = \max_{k \in R} \sum_{p=1}^{H+1} X_{jk}^p$ holds. Let $(\mathbf{X}', \mathbf{x}', \mathbf{z}')$ be defined as: $z_j' := z_j$ for all j in F; ${X'}_{ij}^p := X_{ij}^p$ for all ij in A_S, $p \in \mathbf{I}_1^H$ and $x'_{jk} := \sum_{p=1}^{H+1} X_{jk}^p$ for all jk in A_R. Then, $(\mathbf{X}', \mathbf{x}', \mathbf{z}')$ has the same objective value as $(\mathbf{X}, \mathbf{z})$. To

conclude this part of the proof, we will show that it is a feasible solution from $\mathcal{P}(HOP_F)$. Inequalities (10a) and (10d) follow from (9a) and (9e). Constraints (1a) and (1b) are implied by (9c) and (9b) respectively. For all $j \in F \setminus \{r\}$ let $k^j := \arg\max_{k \in R} \sum_{p=2}^{H+1} X_{jk}^p$. Then, w.l.o.g., we have $z_j = \sum_{p=2}^{H+1} X_{jk^j}^p$ and further we have

$$z_j' = z_j = \sum_{p=2}^{H+1} X_{jk^j}^p \overset{(9a)}{\le} \sum_{p=1}^{H} \sum_{ij \in A_S} X_{ij}^p = \sum_{p=1}^{H} \sum_{ij \in A_S} X'^p_{ij},$$

hence equations (10b) also hold for $(\mathbf{X}', \mathbf{x}', \mathbf{z}')$.

$\upsilon_{LP}(HOP_F) \ge \upsilon_{LP}(HOP_R)$: Let $(\mathbf{X}', \mathbf{x}', \mathbf{z}') \in \mathcal{P}(HOP_F)$ and $(\mathbf{X}, \mathbf{z})$ defined as $z_j := z_j'$ for all j in F and $X_{ij}^p :=$ X'^p_{ij} for all ij in A_S, $p \in \mathbf{I}_1^H$. Let $x_{jk}' > 0$ with $j \in F \setminus \{r\}, k \in R$. From equations (10b) and (1b) we have $x_{jk}' \le z_j' \le \sum_{ij \in A_S} \sum_{p=1}^{H} X'^p_{ij}$. From the right hand side we can select X_{ij}^{*p} with $X_{ij}^{*p} \le X'^p_{ij}$ such that $\sum_{ij \in A_S} \sum_{p=1}^{H} X_{ij}^{*p} = x_{jk}'$. We can do so for all $jk \in A_R$. Let

$$X_{jk}^{p+1} := \sum_{ij \in A_S} X_{ij}^{*p} \qquad \forall j \in F \setminus \{r\}, k \in R, \ p \in \mathbf{I}_1^H \tag{11}$$

and $X_{rk}^1 := x_{rk}'$ for all $k \in R$. Then we can show that $(\mathbf{X}, \mathbf{z}) \in \mathcal{P}(HOP_R)$: Equations (9a) follow from the definitions. Constraints (9c) follow from

$$\sum_{p=1}^{H+1} X_{jk}^p \overset{(10d)}{=} 0 + \sum_{p=2}^{H+1} X_{jk}^p \overset{(11)}{=} \sum_{ij \in A_S} \sum_{p=1}^{H} X_{ij}^{*p} = x_{jk}' \le z_j' = z_j \qquad \forall jk \in A_R.$$

Constraints (9b) follow from

$$\sum_{p=1}^{H+1} \sum_{jk \in A_R} X_{jk}^p = X_{rk}^1 + \sum_{p=1}^{H} \sum_{ij \in A_S} X_{ij}^{*p} = X_{rk}^1 + \sum_{ij \in A_S} \sum_{p=1}^{H+1} X_{ij}^{*p} = \sum_{jk \in A_R} x_{jk}' \overset{(1a)}{=} 1.$$

$\square$

2.5. Modeling Hop Constraints on Layered Graphs

In [3] we have proposed several ways of modelling HC ConFL as ConFL or Steiner arborescence problem on layered graphs. We recite the main results given there. In Section 3 we prove new relations in addition to the ones stated in our earlier work.

There are two basic structures of layered graphs. In the *customer based approach*, we disaggregate both the core and assignment graph by introducing a copy of each node in the core graph at each level h, $1 \le h \le H$. Customers can be assigned to each facility at each level via a distinct arc. In the *facility based approach*, copies of facilities at a lower level than H are considered as Steiner nodes, thus the assignment graph is identical to the one in the original graph.

2.5.1. Layered Core and Assignment Graph $LG_{x,z}$ (Customer Based Approach)

Consider a graph $LG_{x,z} = (V_{x,z}, A_{x,z})$ defined as follows:

$$V_{x,z} := \{r\} \cup S_{x,z} \cup R \text{ where}$$
$$F_{x,z} = \{(i, p) : i \in F \setminus \{r\}, p \in \mathbf{I}_1^H\},$$
$$S_{x,z} = F_{x,z} \cup \{(i, p) : p \in \mathbf{I}_1^{H-1}, i \in S \setminus F\} \text{ and}$$
$$A_{x,z} := \bigcup_{i=1}^{5} A_i \text{ where}$$
$$A_1 = \{(r, (j, 1)) : rj \in A_S\},$$
$$A_2 = \{((i, p), (j, p+1)) : p \in \mathbf{I}_1^{H-2}, ij \in A_S\},$$
$$A_3 = \{((i, H-1), (j, H)) : ij \in A_S, i \in S \setminus \{r\}, j \in F \setminus \{r\}\},$$
$$A_4 = \{rk : rk \in A_R\}$$
$$A_5 = \{((i, p), k) \mid ik \in A_R, (i, p) \in F_{x,z}, k \in R\}.$$

In [3] we have shown that one can obtain the optimal solution of HC ConFL by solving ConFL on $LG_{x,z}$ with $F_{x,z}$ as the set of facilities. The cost structure on $LG_{x,z}$ is as follows: Arcs in $A_1 \cup A_2 \cup A_3$ and $A_4 \cup A_5$ are assigned the cost of the corresponding arc from A_S and A_R, respectively. The facility opening costs are f_i for all (i, p) with $p \in \mathbf{I}_1^H$, $i \in F \setminus \{r\}$.

An illustration of $LG_{x,z}$ for the example in Figure 1 is shown in Figure 5a).

We will associate binary variables to the arcs in $A_{x,z}$ as follows: X_{rj}^1 corresponds to $(r, (j, 1)) \in A_1$, X_{ij}^p to $((i, p-1), (j, p)) \in A_2$, X_{ij}^H to $((i, H-1), (j, H)) \in A_3$, X_{rk}^1 to $rk \in A_4$ and X_{ik}^{p+1} corresponds to $((i, p), k) \in A_5$. For notational convenience we also introduce the following variables: a) X_{ri}^p, for $ri \in A_S$, $p \in \mathbf{I}_2^H$; b) X_{rk}^p, for $rk \in A_R$, $p \in \mathbf{I}_2^{H+1}$; c) X_{ij}^1 for $ij \in A$, $i \neq r$ and d) X_{ij}^H for $ij \in A_S$, $j \in S \setminus F$. These variables are fixed to zero throughout the remainder of this paper.

We link the variables introduced above to the initial ones, $\mathbf{x}$ and $\mathbf{z}$, as follows:

$$x_{ij} = \sum_{p=1}^{H} X_{ij}^p \qquad \forall ij \in A_S \tag{12a}$$

$$x_{jk} = \sum_{p=0}^{H} X_{jk}^{p+1} \qquad \forall jk \in A_R \tag{12b}$$

$$z_j = \sum_{p=1}^{H} Z_j^p \qquad \forall j \in F \setminus \{r\} \tag{12c}$$

Let $X[\delta^-(W)]$ denote the sum of all variables $\mathbf{X}$ in the cut $\delta^-(W)$ in $LG_{x,z}$ defined by $W \subseteq V_{x,z} \setminus \{r\}$. Model $CUT_{F+}^{x,z}$ is then given by the following:

$$(CUT_{F+}^{x,z}) \quad \min f(x, z) = \sum_{ij \in A} c_{ij} x_{ij} + \sum_{j \in F} f_j z_j$$

$$\text{s.t.} \quad X[\delta^-(W)] \geq \sum_{p=1}^{H} Z_i^p \qquad \forall W : \{(i, p) \mid p \in \mathbf{I}_1^H\} \subseteq W \subseteq S_{x,z} \setminus \{r\} \tag{13a}$$

$$0 \leq X_{jk}^{p+1} \leq Z_j^p \qquad \forall jk \in A_R, \ p \in \mathbf{I}_1^H, \ j \neq r \tag{13b}$$

$$X_{ij}^p \geq 0 \qquad \forall ij \in A_S, \ p \in \mathbf{I}_1^H \tag{13c}$$

$$(1a), (1c) - (1f), (12)$$

Constraints (13a) are *connectivity cuts* on $LG_{x,z}$ between the root r and each set of facilities $\{(i, p) \in F_{x,z} \mid p \in \mathbf{I}_1^H\}$. Inequalities (13b) are *coupling constraints* - they necessitate a facility j at a level p to be open if a customer is assigned to it.

If we replace (13a) and (1a) in model $CUT_{F+}^{x,z}$ by the following inequalities we obtain formulation $CUT_R^{x,z}$:

$$X[\delta^-(W)] \geq 1 \qquad \forall W \subseteq V_{x,z} \setminus \{r\}, W \cap R \neq \emptyset, \tag{14}$$

Inequalities (14) are connectivity cuts on $LG_{x,z}$ between sets containing the root and a customer respectively.

2.5.2. Layered Core Graph LG_x

An alternative way of building a layered graph to model the HC ConFL problem relies on a layered graph in which only the core network will be disaggregated while the assignment graph will be left unchanged. Consider the graph $LG_x = (V_x, A_x)$ defined as follows:

$$V_x := \{r\} \cup S_x \cup R \text{ where}$$
$$F_x = \{(i, H) : i \in F \setminus \{r\}\},$$
$$S_x = F_x \cup \{(i, p) : 1 \leq p \leq H - 1, i \in S \setminus \{r\}\} \text{ and}$$
$$A_x := \bigcup_{i=1}^{4} A_i \cup A_6 \cup A_7 \text{ where}$$
$$A_1, A_2, A_3 \text{ and } A_4 \text{ are defined as for } A_{x,z},$$
$$A_6 = \{((i, p), (i, H)) : 1 \leq p \leq H - 1, i \in F \setminus \{r\}\} \text{ and}$$
$$A_7 = \{((j, H), k) : jk \in A_R, j \neq r\}$$

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An illustration of LG_x for the example in Figure 1 is shown in Figure 5b).
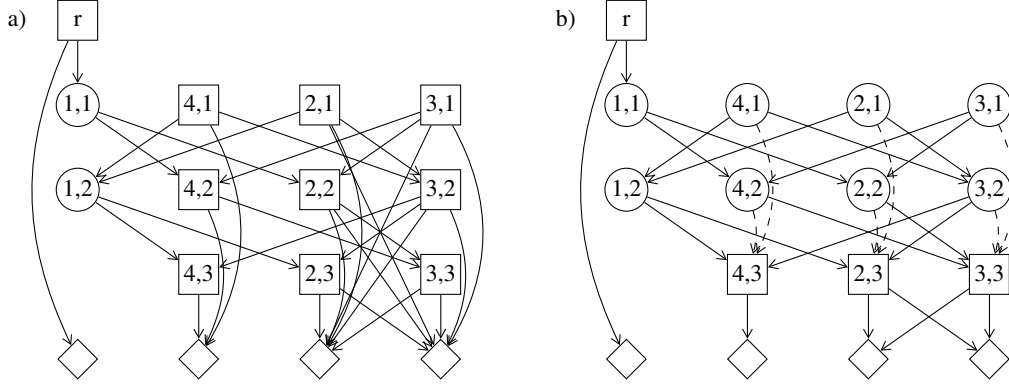

Figure 5: Layered graph a) LG_{xz} and b) LG_x for the instance given in Figure 1a).

The facility opening and assignment costs are left unchanged. The set $A_{S_x} := A_1 \cup A_2 \cup A_3 \cup A_6$ determines the *layered core graph*. The cost of an arc from $A_1 \cup A_2 \cup A_3$ and $A_4 \cup A_7$ is set to the cost of the corresponding arc from A_S and A_R, respectively. Arcs between (i, p) and (i, H) are assigned costs of 0 for all $p \in \mathbf{I}_1^{H-1}$ and $i \in F$.

We will associate binary variables to the arcs in A_x as follows: X_{rj}^1 corresponds to $(r, (j, 1)) \in A_1$, X_{ij}^p to $((i, p-1), (j, p)) \in A_2$, X_{ij}^H to $((i, H-1), (j, H)) \in A_3$, X_{ii}^p to $((i, p-1), (i, H)) \in A_6$. Again, for notational convenience, we also introduce the following binary variables but fix them to zero: a) X_{ri}^p, for $ij \in A_S$, $p \in \mathbf{I}_2^H$, and b) X_{ij}^1, for $ij \in A_S$, $i \neq r$. Since the assignment graph is left unchanged, we will associate the corresponding $\mathbf{x}$ variables to the assignment graph in LG_x, i.e.: x_{jk} to $((j, H), k) \in A_7$ and x_{rk} to $rk \in A_4$. For the same reason, we link binary variables z_i to each (i, H) in F_x.

Let $X_x[\delta^-(W)]$ denote the sum of all $\mathbf{X}$ and $\mathbf{x}$ variables in the cut $\delta^-(W)$ in LG_x defined by $W \subseteq V_x \setminus \{r\}$. Formulation CUT_F^x is then given as follows:

$$(CUT_F^x) \quad \min f(x, z) = \sum_{ij \in A} c_{ij} x_{ij} + \sum_{j \in F} f_j z_j$$

$$X_x[\delta^-(W)] \geq z_j \qquad \forall W \subseteq S_x \setminus \{r\}, \ j \in W \cap F_x \tag{15}$$

$$(1) \text{ and } (12a)$$

Constraints (15) are connectivity cuts on LG_x between sets containing the root and a facility i respectively.

Similarly, if we now replace constraints (15) and (1a) by the following ones, we obtain a stronger formulation that we denote by CUT_R^x:

$$X_x[\delta^-(W)] \geq 1 \quad \forall W \subseteq V_x \setminus \{r\}, W \cap R \neq \emptyset \tag{16}$$

Lemma 10 (Ljubić and Gollowitzer [3]). $v_{LP}(CUT_R^x) \geq v_{LP}(CUT_F^x)$. *Furthermore, there exist HC ConFL instances for which the strict inequality holds.*

Lemma 11. $v_{LP}(CUT_R^{x,z}) = v_{LP}(CUT_R^x)$ *and* $v_{LP}(CUT_{F+}^{x,z}) = v_{LP}(CUT_F^x)$.

Proof. The proof of the first equality is given in [3]. For the proof of the second equality we show mappings for an optimal LP-solution of $CUT_{F+}^{x,z}$ onto a feasible solution of CUT_F^x and vice versa.

$v_{LP}(CUT_{F+}^{x,z}) \geq v_{LP}(CUT_F^x)$: Let $(\mathbf{X}, \mathbf{Z})$ be an optimal LP-solution of the model $CUT_{F+}^{x,z}$. We project $(\mathbf{X}, \mathbf{Z})$ into a solution $(\mathbf{X}', \mathbf{x}', \mathbf{z}')$ and show that it is feasible for the model CUT_F^x. We set $X_{ij}'^p := X_{ij}^p$ for all arcs in A_1, A_2 and A_3; $X_{jj}'^p := Z_j^p (= \max_{k \in R} X_{jk}^p)$ for all arcs in A_6; $x'_{jk} := \sum_{p=1}^H X_{jk}^{p+1}$ for all arcs in A_7; $x'_{rk} := X_{rk}^1$ for all arcs in A_4; $z_j := \sum_{p=1}^H Z_j^p$. All remaining $\mathbf{X}'$ values are set to zero. Inequalities (13b), summed up over p, imply inequalities (1b). Constraints (15) hold by the definition of variables z' and $X_{jj}'^p$.

$v_{LP}(CUT_F^x) \geq v_{LP}(CUT_{F+}^{x,z})$: Let $(\mathbf{X}', \mathbf{x}', \mathbf{z}')$ be an optimal LP-solution of the model CUT_F^x. We project this vector into $(\mathbf{X}, \mathbf{Z})$ as follows: $X_{ij}^p := X_{ij}'^p$ for all arcs in A_1, A_2 and A_3; $X_{rk}^1 := x'_{rk}$ for all arcs in A_4. Furthermore, we set $Z_j^p := X_{jj}'^p$, for all arcs from A_6, for $p \in \mathbf{I}_1^{H-1}$, and $Z_j^H := z'_j - \sum_{p=1}^{H-1} Z_j^p$, for all $j \in F \setminus \{r\}$. We then recursively define $X_{jk}^{p+1} := \min(Z_j^p, x'_{jk} - \sum_{q=p+1}^H X_{jk}^{q+1})$ starting from $p = H, \dots, 1$. By definition, $(\mathbf{X}, \mathbf{Z})$ satisfies constraints (12). $\qquad \square$

3. Hierarchy of formulations

In this section we provide a theoretical comparison of the MIP models described above with respect to optimal values of their LP-relaxations. The examples given below are used in the proofs of this section. In the example in Figure 6 c) $H = 5$, in all other examples $H = 3$. Recall, that the default arc, facility opening and assignment costs are set to 1.
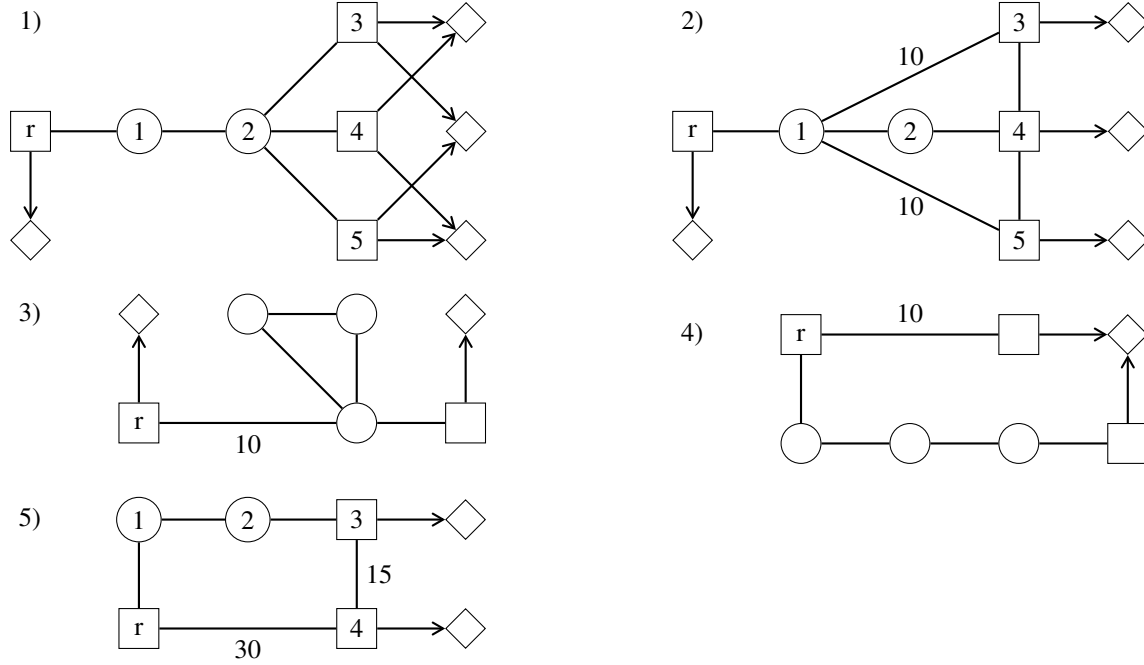

Figure 6: Examples 1 - 5

	Ex. 1	Ex. 2	Ex. 3	Ex. 4	Ex. 5
OPT	29.00	30.00	15.00	13.00	33.00
MTZ	18.00	13.93̇	8.00	7.50	20.25
$HOP_{\{F,R\}}$	18.00	30.00	10.33̇	13.00	33.00
CUT_F^P	18.00	19.80	15.00	8.50	27.00
CUT_R^P	28.00	19.80	15.00	8.50	27.00
MCF_F	18.00	21.00	15.00	8.50	23.00
MCF_F^+	18.00	21.00	15.00	13.00	23.00
MCF_R	28.00	21.00	15.00	9.00	23.00
HD_F	18.00	30.00	15.00	13.00	33.00
HD_R	28.00	30.00	15.00	13.00	33.00

Table 1: Optimal LP solutions for the examples in Figure 6.

By comparing the optimal LP solution values for the aforementioned examples, provided by the models in Section 2, we can state the following

Lemma 12. *In the following pairs of formulations neither of the two dominates the other with respect to the quality of lower bounds:*

a) $HOP_.$ and $CUT_.^P$,

b) $HOP_.$ and $MCF_.^{(+)}$, where $\cdot$ is to be replaced by F or R and

c) MCF_F^+ and MCF_R.

d) $MCF_.^{(+)}$ and $CUT_.^P$

Proof. Consider the optimal LP solution values for the Examples given in Table 1.   □

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Lemma 13. *The following results hold:*

a) $\upsilon_{LP}(CUT_R^x) \ge \upsilon_{LP}(HD_R) \ge \upsilon_{LP}(MCF_R)$,

b) $\upsilon_{LP}(CUT_F^x) \ge \upsilon_{LP}(HD_F) \ge \upsilon_{LP}(MCF_F^+) \ge \upsilon_{LP}(MCF_F)$,

c) $\upsilon_{LP}(HD_R) \ge \upsilon_{LP}(CUT_R^J) \ge \upsilon_{LP}(CUT_R^P)$,

d) $\upsilon_{LP}(HD_F) \ge \upsilon_{LP}(CUT_F^J) \ge \upsilon_{LP}(CUT_F^P)$ *and*

e) $\upsilon_{LP}(HD_F) \ge \upsilon_{LP}(HOP_F) \ge \upsilon_{LP}(MTZ)$.

Proof. Strict inequalities in a) to e) follow from the optimal LP solution values in Table 1.

a) $\upsilon_{LP}(CUT_R^x) \ge \upsilon_{LP}(HD_R)$**:** By disaggregating constraints (7e) and introducing variables X_{ij}^p, formulation HD_R becomes a multi-commodity flow formulation for ConFL on the layered graph LG_x, and hence, equivalent to CUT_R^x. Therefore, CUT_R^x is at least as strong as HD_R.

$\upsilon_{LP}(HD_R) \ge \upsilon_{LP}(MCF_R)$**:** HD_R is a disaggregation of MCF_R ($f_{ij}^k := \sum_{p=1}^{H+1} f_{ij}^{kp}$).

b) The inclusions follow from similar arguments as those used in a), except for the following:

$\upsilon_{LP}(HD_F) \ge \upsilon_{LP}(MCF_F^+)$**:** MCF_F^+ is an aggregation of HD_F ($g_{ij}^k := \sum_{p=1}^H g_{ij}^{kp}$, for all $ij \in A_S$), except for constraints (3). These are implied by the flow conservation constraints (6a)-(6c) together with the fact that variables g_{ij}^{kp} are defined only for $p \in \mathbf{I}_1^H$, thus there is no path between r and any facility with more than H arcs.

c) $\upsilon_{LP}(HD_R) \ge \upsilon_{LP}(CUT_R^J)$**:** Assume that $(\mathbf{x}, \mathbf{z})$ is an optimal LP solution from $\mathcal{P}(HD_R)$ and $(\mathbf{x}, \mathbf{z}) \notin \mathcal{P}(CUT_R^J)$. Then there exists a $(H+1)$-jump J such that $\sum_{ij \in J} x_{ij} = 1 - \epsilon$, and $\epsilon > 0$. Because of the flow preservation constraints (7a) - (7c) there needs to be a flow of ϵ on the path $P = \{ij : i \in S_i, j \in S_{i+1}, i = 0, \ldots, H+1\}$. This flow uses $H+2$ hops and cannot be composed of flow variables f_{ij}^{kp}, $p \in \mathbf{I}_1^{H+1}$, which is a contradiction.

$\upsilon_{LP}(CUT_R^J) \ge \upsilon_{LP}(CUT_R^P)$**:** Assume that $(\mathbf{x}, \mathbf{z}) \in \mathcal{P}_{x,z}(CUT_R^J)$ is optimal and $(\mathbf{x}, \mathbf{z}) \notin \mathcal{P}_{x,z}(CUT_R^P)$. Let $P = \{(r, i_1), (i_1, i_2), \ldots, (i_H, i_{H+1})\}$ with $i_{H+1} \in F$ be the path for which constraint (P1) is violated, i.e. inequality $\sum_{ij \in P}(x_{ij} + x_{ji}) > H$ holds. For the jump J with $S_j = \{i_j\}$ for $j = 1, \ldots, H+1$, $S_{H+2} = \{k\}, k \in R$ and $S_0 = V \setminus \bigcup_{j=1}^{H+2} S_j$ we have $\sum_{ij \in J} x_{ij} \ge 1$. Let $P' := P \cup \{(i_{H+1}, k)\}$. By adding (P1) for P and the jump constraint for J we get $\sum_{j:ij \in P'} x(\delta^-(j)) \ge \sum_{ij \in J \cup P} x_{ij} > H+1$, thus $x(\delta^-(j)) > 1$ for some $j \in \{i_1, \ldots, i_{H+1}\}$, which is a contradiction to $(\mathbf{x}, \mathbf{z})$ being an optimal LP solution. In a similar way one can show that the jump formulation for HCSTP is stronger than the path formulation [21].

d) See the arguments in c). The sum of the in-degrees of the nodes in P' is calculated as follows:
$\sum_{j:ij \in P'} x(\delta^-(j)) \ge \sum_{ij \in P} x_{ij} + \sum_{ij \in J} x_{ij} + \sum_{i \in F \setminus \{j\}} x_{ik} > H + z_j + \sum_{i \in F \setminus \{j\}} x_{ik} \ge H + \sum_{i \in F} x_{ik} = H + 1.$

e) $\upsilon_{LP}(HD_F) \ge \upsilon_{LP}(HOP_F)$ Let $(\mathbf{x}, \mathbf{z}, \mathbf{f})$ be an optimal solution for the LP-relaxation of HD_F and let $(\mathbf{X}', \mathbf{x}', \mathbf{z}')$ be defined as follows: $x'_{jk} := x_{jk}$ for all jk in A_R; $z'_j := z_j$ for all j in F and $X'^p_{ij} := \max_{k \in F} f_{ij}^{kp}$ for all ij in A_S, $p \in \mathbf{I}_1^H$.
Then $(\mathbf{X}', \mathbf{x}', \mathbf{z}') \in \mathcal{P}(HOP_F)$: From equations (6c) and the definition of X' we have

$$z_j = \sum_{p=1}^H \sum_{ij \in A_S} f_{ij}^{jp} \le \sum_{p=1}^H \sum_{ij \in A_S} \max_{k \in F} f_{ij}^{kp} = \sum_{p=1}^H \sum_{ij \in A_S} X'^p_{ij}$$

for compliance with equations (10b). With $k^* := \arg\max_{k \in F} f_{ij}^{kp}$ estimations

$$X'^p_{ij} = \max_{k \in F} f_{ij}^{kp} \overset{(6a)}{\le} \sum_{\substack{l \in S \setminus \{j\}: \\ li \in A_S}} f_{li}^{k^*, p-1} \le \sum_{\substack{l \in S \setminus \{j\}: \\ li \in A_S}} \max_{k \in F} f_{li}^{k, p-1} = \sum_{\substack{l \in S \setminus \{j\}: \\ li \in A_S}} X'^{p-1}_{li}$$

give equations (10a). Constraints (1a) - (1c), (1e), (1f) are common in both models and thus they are met trivially.

$\upsilon_{LP}(HOP_F) \ge \upsilon_{LP}(MTZ)$**:** MTZ is an aggregation of HOP_F (cf. [13]; $u_j := \sum_{p=1}^H p X_{ij}^p$).

□

The hierarchical scheme given in Figure 7 summarizes the relationships between the LP relaxations of MIP models considered throughout this paper. A filled arrow specifies that the target formulation is strictly stronger than the source formulation. A double-headed arrow denotes formulations of equal strength. Whenever formulations are not comparable or we do not know their relation, this is not indicated in the figure for the sake of simplicity.
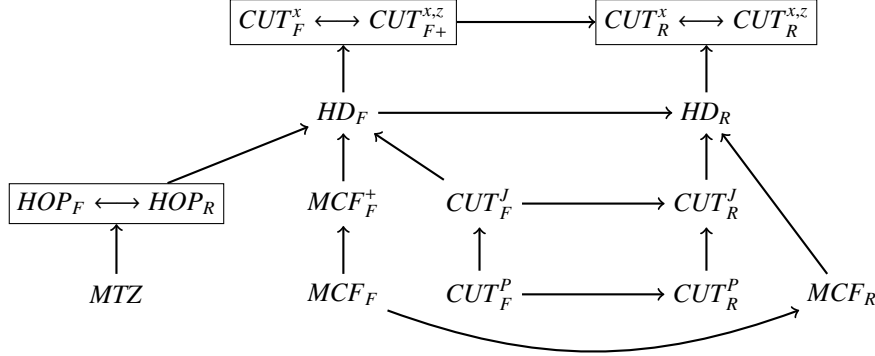

Figure 7: Relations between LP-relaxations of MIP models for HC ConFL.

4. Conclusions and Future Work

In this paper we provide an extensive theoretical comparison of the LP relaxations of 16 MIP models for HC ConFL. We also introduce new sets of path and jump inequalities that can be used to model hop constrained tree problems with node variables in general.

We derive two groups of models for HC ConFL: in the first one, we ensure that the number of hops between the root and any open facility is at most H, in the second one we guarantee that the number of hops between the root and any customer is at most $H + 1$. The hierarchy of formulations shows that the models derived on layered graphs provide the best quality of lower bounds.

In an accompanying paper (see Ljubić and Gollowitzer [22]), we provide a computational study on some of the computationally tractable models presented in this paper. We show that the facility based cut model on the layered graph (CUT_F^x) outperforms its customer based counterpart (CUT_R^x). Among the compact models, only the model HOP_F appears to be able to deal with larger problem instances, but it is still outperformed by the branch-and-cut approaches.

A further study of valid and facet defining inequalities that could improve the computational efficacy remains open and of particular interest as well.

Acknowledgements

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