A Note on the Bertsimas & Sim Algorithm for Robust Combinatorial Optimization Problems

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Abstract

We improve the well-known result of Bertsimas and Sim presented in (D. Bertsimas and M. Sim., "Robust discrete optimization and network flows", *Mathematical Programming*, B(98): 49-71, 2003) regarding the computation of optimal solutions of Robust Combinatorial Optimization problems with interval uncertainty in the objective function coefficients. We also extend this improvement to a more general class of Combinatorial Optimization problems with interval uncertainty.

Keywords: Robust Combinatorial Optimization Problems, Bertsimas & Sim Algorithm.

We address a binary optimization problem on N variables in which both the objective function coefficients (*costs*) and the data in the constraints are subject to uncertainty. When only the costs are subject to uncertainty, a well-known result of Bertsimas and Sim (2003) states that the Bertsimas & Sim robust counterpart, in the remainder simply called *robust counterpart*, of the problem can be solved by solving at most $N+1$ instances of the original deterministic problem.

Thus, the robust counterpart of a polynomially solvable binary optimization problem remains polynomially solvable. For a given *level of conservatism* $0 < \Gamma \leq N$, which is interpreted as the number of coefficients that are expected to present uncertainty, we improve this important result as follows:

1. When only the cost coefficients are subject to uncertainty, the robust counterpart of the problem can be solved by solving at most $N - \Gamma + 2$ instances of the original deterministic problem.
2. When only coefficients of a knapsack constraint in the corresponding mathematical programming problem are subject to uncertainty, the robust counterpart of the problem can be solved by solving at most $N - \Gamma + 2$ instances of the original deterministic problem.
3. We also consider a general case in which the set of variables is partitioned into $K + L$ subsets of size $N_1, \ldots, N_{K+L}$ $(K, L \geq 0,\ K + L \geq 1)$, where the costs of the variables from the first K subsets and also coefficients of L knapsack constraints (associated to variables of remaining subsets) are subject to uncertainty, and to each of the subsets a level of conservatism $0 < \Gamma_k \leq N_k$ and $0 < \Gamma_l \leq N_L$, for $k \in \{1, \ldots, K\}$ and $l \in \{1, \ldots, L\}$ is associated. Then, the robust counterpart of the problem can be solved by solving $\prod_{k=1}^{K}(N_k - \Gamma_k + 2)\prod_{l=1}^{L}(N_l - \Gamma_l + 2)$ instances of the original deterministic problem.

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To prove these results, we will consider a class of binary optimization problems with two types of binary variables, each with its own level of conservatism associated with its corresponding coefficients. The results 1 and 2 are special cases, and the general result stated above follows by mathematical induction.

Let us consider the following generic Combinatorial Optimization problem with linear objective function and two types of binary variables $\mathbf{x} \in \{0,1\}^n$ and $\mathbf{y} \in \{0,1\}^m$:

$$OPT_{P1} = \min \left\{ \sum_{i \in I} c_i x_i + \sum_{j \in J} b_j y_j \mid D\mathbf{x} + E\mathbf{y} \leq \mathbf{f} \text{ and } (\mathbf{x}, \mathbf{y}) \in \Phi \right\}, \tag{P1}$$

where $\mathbf{c}, \mathbf{b}, \mathbf{f} \geq 0$, $I = \{1, 2, \ldots, n\}$, $J = \{1, 2, \ldots, m\}$, D and E are real non-zero matrices and Φ is a generic polyhedral region.

Let us assume now that instead of having known and deterministic parameters c_i $\forall i \in I$ and b_j $\forall j \in J$, we are actually given uncertain intervals $[c_i, c_i + d_i]$ $\forall i \in I$ and $[b_j, b_j + \delta_j]$ $\forall j \in J$. Assume that variables $\mathbf{x}$ and $\mathbf{y}$ are ordered so that $d_i \geq d_{i+1}$ $\forall i \in I$ and $d_{n+1} = 0$, $\delta_j \geq \delta_{j+1}$ $\forall j \in J$ and $\delta_{m+1} = 0$. For simplicity of notation we will assume that $\Gamma_X \in \{1, \ldots, n\}$ and $\Gamma_Y \in \{1, \ldots, m\}$.

For a given pair (Γ_X, Γ_Y), the B&S robust counterpart of this problem is:

$$ROPT_{P1}(\Gamma_X, \Gamma_Y) = \min \left\{ \sum_{i \in I} c_i x_i + \beta_X^* (\Gamma_X, \mathbf{x}) + \sum_{j \in J} b_j y_j + \beta_Y^* (\Gamma_Y, \mathbf{y}) \mid D\mathbf{x} + E\mathbf{y} \leq \mathbf{f} \text{ and } (\mathbf{x}, \mathbf{y}) \in \Phi \right\},$$

where $\beta_X^* (\Gamma_X, \mathbf{x})$ and $\beta_Y^* (\Gamma_Y, \mathbf{y})$ are the corresponding *protection functions* defined as:

$$\beta_X^* (\Gamma_X, \mathbf{x}) = \max \left\{ \sum_{i \in I} d_i x_i u_i \mid \sum_{i \in I} u_i \leq \Gamma_X \text{ and } u_i \in [0,1] \, \forall i \in I \right\} \tag{1}$$

and

$$\beta_Y^* (\Gamma_Y, \mathbf{y}) = \max \left\{ \sum_{j \in J} \delta_j y_j v_j \mid \sum_{j \in J} v_j \leq \Gamma_Y \text{ and } v_j \in [0,1] \, \forall j \in J \right\}. \tag{2}$$

These protection functions provide robustness to the solutions in terms of protection of optimality in presence of a given level of data uncertainty, represented by Γ_X and Γ_Y.

For simplicity of notation let Ω be the set of all the feasible solutions $(\mathbf{x}, \mathbf{y})$ satisfying $D\mathbf{x} + E\mathbf{y} \leq \mathbf{f}$ and $(\mathbf{x}, \mathbf{y}) \in \Phi$. After applying strong duality to (1) and (2), the problem $ROPT_{P1}$ can be rewritten as

$$ROPT_{P1}(\Gamma_X, \Gamma_Y) = \min \sum_{i \in I} c_i x_i + \Gamma_X \theta + \sum_{i \in I} h_i + \sum_{j \in J} b_j y_j + \Gamma_Y \lambda + \sum_{j \in J} k_j \tag{3}$$

subject to

$$h_i + \theta \geq d_i x_i, \; \forall i \in I \tag{4}$$

$$k_j + \lambda \geq \delta_j y_j, \; \forall j \in J \tag{5}$$

$$h_i \geq 0 \; \forall i \in I, \; k_j \geq 0 \; \forall j \in J \text{ and } \theta, \lambda \geq 0 \tag{6}$$

$$(\mathbf{x}, \mathbf{y}) \in \Omega. \tag{7}$$

The following lemma that gives upper bounds for the values of θ and λ is crucial for the results stated in this work.

Lemma 1. *Given* $\Gamma_X \in \{1,\dots,n\}$ *and* $\Gamma_Y \in \{1,\dots,m\}$, *any optimal solution* $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{h}^*, \mathbf{k}^*, \theta^*, \lambda^*)$ *of the robust counterpart of* (P1) *satisfies:* $\theta^* \in [0, d_{\Gamma_X}]$ *and* $\lambda^* \in [0, d_{\Gamma_Y}]$.

Proof. Given the structure of constraints $h_i + \theta \geq d_i x_i$, $\forall i \in I$ and $k_j + \lambda \geq \delta_j y_j$, $\forall j \in J$, it follows that any optimal solution $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{h}^*, \mathbf{k}^*, \theta^*, \lambda^*)$ satisfies:

$$\begin{aligned} h_i^* &= \max\left(d_i x_i^* - \theta^*, 0\right) \\ k_j^* &= \max\left(\delta_j y_j^* - \lambda^*, 0\right), \end{aligned}$$

and since $x_i \in \{0,1\}$ and $y_j \in \{0,1\}$, then it is true that

$$\begin{aligned} \max\left(d_i x_i^* - \theta^*, 0\right) &= \max\left(d_i - \theta^*, 0\right) x_i^* \\ \max\left(\delta_j y_j^* - \lambda^*, 0\right) &= \max\left(\delta_j - \lambda^*, 0\right) y_j^*. \end{aligned}$$

Therefore, the objective function of the problem can be rewritten as

$$ROPT_{P1}(\Gamma_X, \Gamma_Y) = \min \sum_{i \in I} c_i x_i + \Gamma_X \theta + \sum_{i \in I} \max\left(d_i - \theta, 0\right) x_i + \sum_{j \in J} b_j y_j + \Gamma_Y \lambda + \sum_{j \in J} \max\left(\delta_j - \lambda, 0\right) y_j.$$

Let $(\mathbf{x}, \mathbf{y})$ be a feasible solution for a given pair (Γ_X, Γ_Y). Let $N_{\mathbf{x}}$ be the set of indices i such that $x_i = 1\ \forall i \in N_{\mathbf{x}}$ and $x_i = 0$ otherwise. Let $I(N_{\mathbf{x}}, \Gamma_X)$ be a subset of $N_{\mathbf{x}}$ with indices of at most Γ_X elements which have the largest deviations.

Let us assume that $|N_{\mathbf{x}}| \leq \Gamma_X$, then we have $I(N_{\mathbf{x}}, \Gamma_X) = N_{\mathbf{x}}$, which implies that the cost of each element corresponding to an index $i \in N_{\mathbf{x}}$ will be set to its corresponding upper bound $c_i + d_i$, i.e., the minimum value $ROPT_{P1}(\Gamma_X, \Gamma_Y)$ will be reached for $\theta^* = d_{n+1} = 0$.
Let us now assume that $|N_{\mathbf{x}}| \geq \Gamma_X + 1$. Then, by definition, we have $|I(N_{\mathbf{x}}, \Gamma_X)| = \Gamma_X$. Let r^* be the index of the Γ_X-th largest deviation taken into the solution, i.e., $r^* = \max\{i \,|\, i \in I(N_{\mathbf{x}}, \Gamma_X)\}$. Then we have:

$$\begin{aligned} \sum_{i \in N_{\mathbf{x}}} c_i + \sum_{i \in I(N_{\mathbf{x}}, \Gamma_X)} d_i &= \sum_{i \in N_{\mathbf{x}}} c_i + \sum_{\{i \in N_{\mathbf{x}} : i \leq r^*\}} d_i - \sum_{\{i \in N_{\mathbf{x}} : i \leq r^*\}} d_{r^*} + \sum_{\{i \in N_{\mathbf{x}} : i \leq r^*\}} d_{r^*} \\ &= \sum_{i \in N_{\mathbf{x}}} c_i + \sum_{\{i \in N_{\mathbf{x}} : i \leq r^*\}} (d_i - d_{r^*}) + \Gamma_X d_{r^*} \\ &= \sum_{i \in I} c_i x_i + \sum_{i=1}^{r^*} (d_i - d_{r^*}) x_i + \Gamma_X d_{r^*}. \end{aligned}$$

Note that $r^* \geq \Gamma_X$ since $|N_{\mathbf{x}}| \geq \Gamma_X + 1$. Therefore, the minimum value $ROPT_{P1}(\Gamma_X, \Gamma_Y)$ will be reached for $\theta^* = d_r$ where $r \geq \Gamma_X$, and hence, $\theta^* \in [0, d_{\Gamma_X}]$.

By following the same arguments one can also show that $\lambda^* \in [0, d_{\Gamma_Y}]$ for any optimal solution of $ROPT_{P1}(\Gamma_X, \Gamma_Y)$. ■

The following lemma provides an algorithmic scheme to solve $ROPT_{P1}(\Gamma_X, \Gamma_Y)$ by solving a finite number of nominal (deterministic) problems.

Lemma 2. *Given* $\Gamma_X \in \{1,\dots,n\}$ *and* $\Gamma_Y \in \{1,\dots,m\}$, *the robust counterpart of the generic problem (P1) can be solved by solving* $(n - \Gamma_X + 2)(m - \Gamma_Y + 2)$ *nominal problems*

$$ROPT_{P1}(\Gamma_X, \Gamma_Y) = \min_{\substack{r \in \{\Gamma_X, \dots, n+1\} \\ s \in \{\Gamma_Y, \dots, m+1\}}} G^{r,s},$$

where for $r \in \{\Gamma_X, \ldots, n+1\}$ *and* $s \in \{\Gamma_Y, \ldots, m+1\}$*:*

$$G^{r,s} = \Gamma_X d_r + \Gamma_Y \delta_s + \min_{(\mathbf{x},\mathbf{y})\in\Omega} \left(\sum_{i\in I} c_i x_i + \sum_{i=1}^{r} (d_i - d_r)\, x_i + \sum_{j\in J} b_j y_j + \sum_{j=1}^{s} (\delta_j - \delta_s)\, y_j \right).$$

Proof. Using the result of Lemma 1 we can rewrite the robust counterpart of $(P1)$ as

$$ROPT_{P1}(\Gamma_X, \Gamma_Y) = \min \sum_{i\in I} c_i x_i + \Gamma_X \theta + \sum_{i\in I} \max\left(d_i - \theta, 0\right) x_i + \sum_{j\in J} b_j y_j + \Gamma_Y \lambda + \sum_{j\in J} \max\left(\delta_j - \lambda, 0\right) y_j$$

subject to $\theta \le d_{\Gamma_X}$, $\lambda \le d_{\Gamma_Y}$ and (4)-(7).

As it is done in (Bertsimas and Sim, 2003), we find the optimal values of θ and λ by using a decomposition approach. We consider a decomposition of the real interval $[0, d_{\Gamma_X}]$ in $[0, d_n]$, $[d_n, d_{n-1}]$, ..., $[d_{\Gamma_X+1}, d_{\Gamma_X}]$ with respect to the d_i deviations, and a decomposition of the real interval $[0, \delta_{\Gamma_Y}]$ in $[0, \delta_m]$, $[\delta_m, \delta_{m-1}]$, ..., $[\delta_{\Gamma_Y+1}, \delta_{\Gamma_Y}]$ with respect to the δ_j deviations. Observe that for an arbitrary $\theta \in [d_r, d_{r-1}]$ and $\lambda \in [\delta_s, \delta_{s-1}]$ we have:

$$\sum_{i\in I} \max(d_i - \theta, 0) x_i = \sum_{i=1}^{r-1} (d_i - \theta) x_i \quad \text{and} \quad \sum_{j\in J} \max(\delta_j - \lambda, 0) y_j = \sum_{j=1}^{s-1} (\delta_j - \lambda) y_j$$

Therefore, $ROPT_{P1}(\Gamma_X, \Gamma_Y) = \min_{\substack{r\in\{\Gamma_X,\ldots,n+1\} \\ s\in\{\Gamma_Y,\ldots,m+1\}}} G^{r,s}$ where for $r \in \{\Gamma_X, \ldots, n+1\}$ and $s \in \{\Gamma_Y, \ldots, m+1\}$

$$G^{r,s} = \min \sum_{i\in I} c_i x_i + \Gamma_X \theta + \sum_{i=1}^{r-1} (d_i - \theta)\, x_i + \sum_{j\in J} b_j y_j + \Gamma_Y \lambda + \sum_{i=1}^{s-1} (\delta_j - \lambda)\, y_j, \tag{8}$$

where $\theta \in [d_r, d_{r-1}]$, $\lambda \in [\delta_s, \delta_{s-1}]$ and $(\mathbf{x}, \mathbf{y}) \in \Omega$. Since we are optimizing a linear function of θ over the interval $[d_r, d_{r-1}]$ and also a linear function for λ over the interval $[\delta_s, \delta_{s-1}]$, the optimal value of $G^{r,s}$ is obtained for $(\theta, \lambda) \in \{(d_r, \delta_s), (d_{r-1}, \delta_s), (d_r, \delta_{s-1}), (d_{r-1}, \delta_{s-1})\}$. So, for $r \in \{\Gamma_X, \ldots, n+1\}$ and $s \in \{\Gamma_Y, \ldots, m+1\}$:

$$\begin{aligned}
G^{r,s} \;=\; & \min \Bigg[\Gamma_X d_r + \Gamma_Y \delta_s + \min_{(\mathbf{x},\mathbf{y})\in\Omega} \left(\sum_{i\in I} c_i x_i + \sum_{i=1}^{r-1} (d_i - d_r)\, x_i + \sum_{j\in J} b_j y_j + \sum_{j=1}^{s-1} (\delta_j - \delta_s)\, y_j \right), \\
& \Gamma_X d_{r-1} + \Gamma_Y \delta_s + \min_{(\mathbf{x},\mathbf{y})\in\Omega} \left(\sum_{i\in I} c_i x_i + \sum_{i=1}^{r-1} (d_i - d_{r-1})\, x_i + \sum_{j\in J} b_j y_j + \sum_{j=1}^{s-1} (\delta_j - \delta_s)\, y_j \right), \\
& \Gamma_X d_r + \Gamma_Y \delta_{s-1} + \min_{(\mathbf{x},\mathbf{y})\in\Omega} \left(\sum_{i\in I} c_i x_i + \sum_{i=1}^{r-1} (d_i - d_r)\, x_i + \sum_{j\in J} b_j y_j + \sum_{j=1}^{s-1} (\delta_j - \delta_{s-1})\, y_j \right), \\
& \Gamma_X d_{r-1} + \Gamma_Y \delta_{s-1} + \min_{(\mathbf{x},\mathbf{y})\in\Omega} \left(\sum_{i\in I} c_i x_i + \sum_{i=1}^{r-1} (d_i - d_{r-1})\, x_i + \sum_{j\in J} b_j y_j + \sum_{j=1}^{s-1} (\delta_j - \delta_{s-1})\, y_j \right) \Bigg] \\
\;=\; & \min \Bigg[\Gamma_X d_r + \Gamma_Y \delta_s + \min_{(\mathbf{x},\mathbf{y})\in\Omega} \left(\sum_{i\in I} c_i x_i + \sum_{i=1}^{r} (d_i - d_r)\, x_i + \sum_{j\in J} b_j y_j + \sum_{j=1}^{s} (\delta_j - \delta_s)\, y_j \right), \\
& \Gamma_X d_{r-1} + \Gamma_Y \delta_s + \min_{(\mathbf{x},\mathbf{y})\in\Omega} \left(\sum_{i\in I} c_i x_i + \sum_{i=1}^{r-1} (d_i - d_{r-1})\, x_i + \sum_{j\in J} b_j y_j + \sum_{j=1}^{s} (\delta_j - \delta_s)\, y_j \right), \\
& \Gamma_X d_r + \Gamma_Y \delta_{s-1} + \min_{(\mathbf{x},\mathbf{y})\in\Omega} \left(\sum_{i\in I} c_i x_i + \sum_{i=1}^{r} (d_i - d_r)\, x_i + \sum_{j\in J} b_j y_j + \sum_{j=1}^{s-1} (\delta_j - \delta_{s-1})\, y_j \right), \\
& \Gamma_X d_{r-1} + \Gamma_Y \delta_{s-1} + \min_{(\mathbf{x},\mathbf{y})\in\Omega} \left(\sum_{i\in I} c_i x_i + \sum_{i=1}^{r-1} (d_i - d_{r-1})\, x_i + \sum_{j\in J} b_j y_j + \sum_{j=1}^{s-1} (\delta_j - \delta_{s-1})\, y_j \right) \Bigg].
\end{aligned}$$

Therefore,

$$ROPT_{P1}(\Gamma_X, \Gamma_Y) = \min \left[\Gamma_X d_{\Gamma_X} + \Gamma_Y \delta_{\Gamma_Y} + \min_{(\mathbf{x},\mathbf{y})\in\Omega} \left(\sum_{i\in I} c_i x_i + \sum_{j\in J} b_j y_j \right), \ldots, \right.$$

$$\Gamma_X d_r + \Gamma_Y \delta_s + \min_{(\mathbf{x},\mathbf{y})\in\Omega} \left(\sum_{i\in I} c_i x_i + \sum_{i=1}^{r} (d_i - d_r)\, x_i + \sum_{j\in J} b_j y_j + \sum_{j=1}^{s} (\delta_j - \delta_s)\, y_j \right), \ldots,$$

$$\left. \min_{(\mathbf{x},\mathbf{y})\in\Omega} \left(\sum_{i\in I} c_i x_i + \sum_{i\in I} d_i x_i + \sum_{j\in J} b_j y_j + \sum_{j\in J} \delta_j y_j \right) \right],$$

which is what we wanted to prove. ■

Consider now the following variant of (P1), problem (P2):

$$OPT_{P2} = \min \left\{ \sum_{i\in I} c_i x_i \mid \sum_{j\in J} b_j y_j \leq B,\ D\mathbf{x} + E\mathbf{y} \leq \mathbf{f} \text{ and } (\mathbf{x},\mathbf{y}) \in \Phi \right\}, \tag{P2}$$

where B is a constant such that $B \in \mathbb{R}^{\geq 0}$. Given $\Gamma_X \in \{1, \ldots, n\}$ and $\Gamma_Y \in \{1, \ldots, m\}$, the robust counterpart of (P2) is:

$$ROPT_{P2}(\Gamma_X, \Gamma_Y) = \min \left\{ \sum_{i\in I} c_i x_i + \beta_X^* (\Gamma_X, \mathbf{x}) \mid \sum_{j\in J} b_j y_j + \beta_Y^* (\Gamma_Y, \mathbf{y}) \leq B,\ D\mathbf{x} + E\mathbf{y} \leq \mathbf{f} \text{ and } (\mathbf{x},\mathbf{y}) \in \Phi \right\}.$$

In this case, $\beta_Y^* (\Gamma_Y, \mathbf{y})$ provides protection of feasibility in presence of a level of uncertainty given by Γ_Y. This problem can be rewritten as

$$ROPT_{P2}(\Gamma_X, \Gamma_Y) = \min \sum_{i\in I} c_i x_i + \Gamma_X \theta + \sum_{i\in I} h_i$$

subject to

$$\sum_{j\in J} b_j y_j + \Gamma_Y \lambda + \sum_{j\in J} k_j \leq B \tag{9}$$

(4)-(7)

The following lemma extends the result of Lemma 2 for the robust counterpart of (P2).

Lemma 3. *Given* $\Gamma_X \in \{1, \ldots, n\}$ *and* $\Gamma_Y \in \{1, \ldots, m\}$, *any optimal solution* $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{h}^*, \mathbf{k}^*, \theta^*, \lambda^*)$ *of the robust counterpart of* (P2) *satisfies:* $\theta^* \in [0, d_{\Gamma_X}]$ *and* $\lambda^* \in [0, d_{\Gamma_Y}]$.

Proof. From Lemma 2 it follows that $\theta^* \in [0, d_{\Gamma_X}]$. We now show that $\lambda^* \in [0, \delta_{\Gamma_Y}]$.

Constraint (9) can be written as

$$\sum_{j\in J} b_j y_j + \Gamma_Y \lambda + \sum_{j\in J} \max\left(\delta_j - \lambda, 0\right) y_j \leq B. \tag{10}$$

Let $(\mathbf{x}, \mathbf{y})$ be a feasible solution for a given Γ_X and a given Γ_Y. Let $M_\mathbf{y}$ be a set of indices j such that $y_j = 1\ \forall j \in M_\mathbf{y}$ and $y_j = 0$ otherwise. Let $J(M_\mathbf{y}, \Gamma_Y)$ be a subset of $M_\mathbf{y}$ with at most Γ_Y

elements which have the largest deviations. Since $(\mathbf{x}, \mathbf{y})$ is a feasible solution, then the following holds:

$$\sum_{j \in M_{\mathbf{y}}} b_j + \sum_{j \in J(M_{\mathbf{y}}, \Gamma_Y)} \delta_j \leq B.$$

Let us assume that $|M_{\mathbf{y}}| \leq \Gamma_Y$, then we have $J(M_{\mathbf{y}}, \Gamma_Y) = M_{\mathbf{y}}$, which implies that the cost of each element corresponding to index $j \in M_{\mathbf{y}}$ will be set to its corresponding upper bound $b_j + \delta_j$, and hence constraint (10) is satisfied for $\lambda = d_{m+1} = 0$.

Let us now assume that $|M_{\mathbf{y}}| \geq \Gamma_Y + 1$. Then, by definition, we have $|J(M_{\mathbf{y}}, \Gamma_Y)| = \Gamma_Y$. Let $s^* = \max\{j \,|\, j \in J(M_{\mathbf{y}}, \Gamma_Y)\}$. So

$$\begin{aligned}
\sum_{j \in M_{\mathbf{y}}} b_j + \sum_{j \in J(M_{\mathbf{y}}, \Gamma_Y)} \delta_j &= \sum_{j \in M_{\mathbf{y}}} b_j + \sum_{\{j \in M_{\mathbf{y}} : j \leq s^*\}} \delta_j - \sum_{\{j \in M_{\mathbf{y}} : j \leq s^*\}} \delta_{s^*} + \sum_{\{j \in M_{\mathbf{y}} : j \leq s^*\}} \delta_{s^*} \\
&= \sum_{j \in M_{\mathbf{y}}} b_j + \sum_{\{j \in M_{\mathbf{y}} : j \leq s^*\}} (\delta_j - \delta_{s^*}) + \Gamma_Y \delta_{s^*} \\
&= \sum_{j \in J} b_j y_j + \sum_{j=1}^{s^*} (\delta_j - \delta_{s^*}) y_j + \Gamma_Y \delta_{s^*} \leq B.
\end{aligned}$$

Note that $s^* \geq \Gamma_Y$ since $|M_{\mathbf{y}}| \geq \Gamma_Y + 1$, and therefore constraint (9) will be satisfied for all $\lambda = \delta_s$ such that $s \geq \Gamma_Y$. Therefore for any feasible solution we have $\lambda \in [0, \delta_{\Gamma_Y}]$. ■

Lemma 4. *Given $\Gamma_X \in \{1, \ldots, n\}$ and $\Gamma_Y \in \{1, \ldots, m\}$, the robust counterpart of the generic problem* (P2) *can be obtained by solving $(n - \Gamma_X + 2)(m - \Gamma_Y + 2)$ nominal problems, i.e.*

$$ROPT_{P2}(\Gamma_X, \Gamma_Y) = \min_{\substack{r \in \{\Gamma_X, \ldots, n+1\} \\ s \in \{\Gamma_Y, \ldots, m+1\}}} H^{r,s},$$

where for $r \in \{\Gamma_X, \ldots, n+1\}$ and $s \in \{\Gamma_Y, \ldots, m+1\}$:

$$H^{r,s} = \Gamma_X d_r + \min_{(\mathbf{x}, \mathbf{y}) \in \Omega} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r} (d_i - d_r) x_i \,\middle|\, \sum_{j \in J} b_j y_j + \sum_{j=1}^{s} (\delta_j - \delta_s) y_j + \Gamma_Y \delta_s \leq B \right).$$

Proof. Similar to the proof of Lemma 2. ■

The presented results are important when solving robust counterparts of some well-known combinatorial optimization problems in which different levels of conservatism are associated to disjoint subsets of binary variables. For example, in Prize-Collecting Problems (e.g., TSP, Steiner Trees), binary variables are associated to edges and nodes of a graph, and we might associate different levels of conservatism to their corresponding coefficients, costs and prizes, respectively. Other prominent examples include: facility location problems, where location and assignment decisions need to be taken, or vehicle routing problems, involving routing and assignment decision variables.

References

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