A Note on the Bertsimas & Sim Algorithm for Robust Combinatorial Optimization Problems

Eduardo Álvarez-Miranda · Ivana Ljubić · Paolo Toth

Abstract We improve the well-known result of Bertsimas and Sim presented in (D. Bertsimas and M. Sim., "Robust discrete optimization and network flows", *Mathematical Programming*, B(98): 49-71, 2003) regarding the computation of optimal solutions of Robust Combinatorial Optimization problems with interval uncertainty in the objective function coefficients. We also extend this improvement to a more general class of Combinatorial Optimization problems with interval uncertainty.

1. Introduction and Motivation

We address a general class of Combinatorial Optimization problems in which both the objective function coefficients and the constraint coefficients are subject to interval uncertainty. When uncertainty has to be taken into consideration, Robust Optimization (RO) arises as methodological alternative to deal with it. The Bertsimas & Sim Robust (B&S) Optimization approach, introduced in [Bertsimas and Sim(2003)], is one of the most important approaches devised to incorporate this

E. Álvarez-Miranda, P. Toth
DEIS, Università di Bologna
Viale Risorgimento 2, 40136 Bologna, Italy
E-mail: {e.alvarez,paolo.toth}@unibo.it

I. Ljubić
Department of Statistics and Operations Research, Faculty of Business, Economics, and Statistics, University of Vienna
Brünnerstraße 72 A-1210 Vienna, Austria
Tel: +43-1-4277-38661
Fax: +43-1-4277-38699
E-mail: ivana.ljubic@univie.ac.at

type of uncertainty into the decision process. By means of *protection functions*, the obtained solutions are endowed with protection, i.e., they are *robust*, in terms of feasibility and/or optimality for a given *level of conservatism* denoted by a parameter Γ_X, defined by the decision maker. When the coefficients associated with a set of n variables are subject to uncertainty, the *level of conservatism* is interpreted as the number of coefficients that are expected to present uncertainty, i.e., $0 < \Gamma_X \leq n$.

For the case that the uncertain coefficients are only present in the objective function, a well-known result of [Bertsimas and Sim(2003)] states that the robust counterpart of the problem can be computed by solving at most $n + 1$ instances of the original deterministic problem. Thus, the robust counterpart of a polynomially solvable binary optimization problem remains polynomially solvable.

Our Contribution In this paper we propose some improvements and extensions to the algorithmic result presented in [Bertsimas and Sim(2003)]. For the case studied in their paper, we show that instead of solving $n + 1$ deterministic problems, the robust counterpart can be computed by solving $n - \Gamma_X + 2$ deterministic problems (Lemma 1); this improvement is particularly interesting for those cases for which a high level of conservatism, i.e., a large value of Γ_X, is suitable. Additionally, we show that if a knapsack-type constraint is part of a problem and m of its coefficients are affected by uncertainty, an equivalent algorithmic approach can be applied, and the robust counterpart can be computed by solving $m - \Gamma_Y + 2$ deterministic problems (Lemma 2), for $0 < \Gamma_Y \leq m$. Likewise, we show that if the uncertain coefficients in the objective function are associated with two disjoint sets of variables, of size n and m respectively, the robust problem can be computed by solving of $(n - \Gamma_X + 2)(m - \Gamma_Y + 2)$ deterministic problems (Lemma 3), giving to the decision maker the flexibility to define different levels of conservatism to different sets of uncertain parameters. A similar result is also shown for the case that uncertainty is present in a set of n objective function coefficients and in a set of m coefficients of a knapsack-type constraint (Lemma 4). Combining the previous results, we provide a more general result which considers the case in which the uncertain coefficients in the objective function are associated with K disjoint sets of variables and there are L knapsack-type constraints (each of them involving a different set of variables) with uncertain coefficients. For this type of problems, we show that the robust counterpart can be computed by solving a strongly-polynomial number of deterministic problems (Theorem 1).

The presented results are important when solving robust counterparts of some well-known combinatorial optimization problems in which different levels of conservatism are associated to disjoint subsets of binary variables. For example, in *Prize-Collecting Network Design Problems* (PCNDPs) (e.g., TSP, Steiner Trees), binary variables are associated to edges and nodes of a graph, and we might associate different levels of conservatism to their corresponding coefficients, costs and prizes, respectively. Besides defining the objective function as the sum of edge costs and node prizes, PCNDPs are frequently modeled using knapsack-type Budget or Quota constraints, and our results can be used in these cases as well, when the coefficient of these constraints are subject to interval uncertainty.

Similarly, in Facility Location problems, location and allocation decisions need to be taken. Each of these decisions involves disjoint sets of variables and, possibly uncertain, coefficients. In these conditions, different levels of conservatism might be suitable for different sets of coefficients. Other prominent examples of problems within this framework are generalized vehicle routing problems, involving routing, assignment, location, inventory decision variables and more; for which the presented result can be used for solving the corresponding robust counterparts.

The viability of the proposed methods strongly relies on the efficacy to solve the deterministic counterparts.

2.   Main Results

Let us consider the following generic Combinatorial Optimization problem with linear objective function and binary variables $\mathbf{x} \in \{0,1\}^n$:

$$OPT_{P1} = \min \left\{ \sum_{i \in I} c_i x_i \mid \mathbf{x} \in \Pi \right\}, \tag{P1}$$

where $\mathbf{c} \geq 0$, $I = \{1, 2, \ldots, n\}$ and Π is a generic polyhedral region.

Let us assume now that instead of having known and deterministic parameters c_i, $\forall i \in I$, we are actually given uncertain intervals $[c_i, c_i + d_i]$, $\forall i \in I$. Assume that variables $\mathbf{x}$ are ordered so that $d_i \geq d_{i+1}$, $\forall i \in I$, and $d_{n+1} = 0$.

For a given *level of conservatism* $\Gamma_X \in \{1, \ldots, n\}$, the robust formulation of (P1) is defined in [Bertsimas and Sim(2003)] as:

$$ROPT_{P1}(\Gamma_X) = \min \left\{ \sum_{i \in I} c_i x_i + \beta_X^* \left(\Gamma_X, \mathbf{x} \right) \mid \mathbf{x} \in \Pi \right\}, \tag{RP1}$$

where $\beta_X^* \left(\Gamma_X, \mathbf{x} \right)$ is the corresponding *protection function* defined as:

$$\beta_X^* \left(\Gamma_X, \mathbf{x} \right) = \max \left\{ \sum_{i \in I} d_i x_i u_i \mid \sum_{i \in I} u_i \leq \Gamma_X \text{ and } u_i \in [0,1] \, \forall i \in I \right\}. \tag{1}$$

This protection function endows robustness to the solutions in terms of protection of optimality in presence of a given level of data uncertainty, represented by Γ_X.

In the context of RO, (P1) is referred to as the *nominal problem* and (RP1) as the corresponding *robust counterpart*.

After applying strong duality to (1), problem (RP1) can be rewritten as

$$ROPT_{P1}(\Gamma_X) = \min \sum_{i \in I} c_i x_i + \Gamma_X \theta + \sum_{i \in I} h_i \tag{2}$$

$$\text{s.t.} \quad h_i + \theta \geq d_i x_i, \quad \forall i \in I \tag{3}$$

$$h_i \geq 0, \, \forall i \in I \text{ and } \theta \geq 0 \tag{4}$$

$$\mathbf{x} \in \Pi. \tag{5}$$

The previous formulation of the robust counterpart of (P1) has been presented in [Bertsimas and Sim(2003)] and the authors provide a combinatorial algorithm that computes $ROPT_{P1}(\Gamma_X)$ by solving $n+1$ nominal problems (Theorem 3, p. 56). The following lemma provides an improvement to this result by reducing the number of iterations of the algorithmic procedure.

Lemma 1 *Given $\Gamma_X \in \{1,\ldots,n\}$, the problem* (RP1)*, the robust counterpart of problem* (P1)*, can be computed by solving $(n - \Gamma_X + 2)$ nominal problems in the following scheme:*

$$ROPT_{P1}(\Gamma_X) = \min_{r \in \{\Gamma_X,\ldots,n+1\}} G^r,$$

where for $r \in \{\Gamma_X,\ldots,n+1\}$:

$$G^r = \Gamma_X d_r + \min_{\mathbf{x} \in \Pi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r} (d_i - d_r) x_i \right).$$

Proof. The first part of the proof consists of showing that any optimal solution $(\mathbf{x}^*, \mathbf{h}^*, \theta^*)$ of (RP1) satisfies: $\theta^* \in [0, d_{\Gamma_X}]$.

Given the structure of constraints $h_i + \theta \geq d_i x_i, \forall i \in I$, it follows that any optimal solution $(\mathbf{x}^*, \mathbf{h}^*, \theta^*)$ satisfies:

$$h_i^* = \max(d_i x_i^* - \theta^*, 0),$$

and since $x_i \in \{0, 1\}$, then it is true that

$$\max(d_i x_i^* - \theta^*, 0) = \max(d_i - \theta^*, 0) x_i^*.$$

Therefore, the objective function of (2)-(5) can be rewritten as

$$ROPT_{P1}(\Gamma_X) = \min \sum_{i \in I} c_i x_i + \Gamma_X \theta + \sum_{i \in I} \max(d_i - \theta, 0) x_i.$$

Let $\mathbf{x}$ be a feasible solution for a given Γ_X. Let $N_{\mathbf{x}}$ be the set of indices $i \in I$ such that $x_i = 1$. Let $I(N_{\mathbf{x}}, \Gamma_X)$ be a subset of $N_{\mathbf{x}}$ associated with (at most) the Γ_X largest d_i values.

Let us assume that $|N_{\mathbf{x}}| \leq \Gamma_X$, then we have $I(N_{\mathbf{x}}, \Gamma_X) = N_{\mathbf{x}}$, which implies that the cost of each element corresponding to an index $i \in N_{\mathbf{x}}$ will be set to its corresponding upper bound $c_i + d_i$. This means that if $\mathbf{x}$ is optimal, the minimum value $ROPT_{P1}(\Gamma_X)$ can be calculated as $\sum_{i \in N_{\mathbf{x}}}(c_i + d_i)$, which implies that $\theta^* = d_{n+1} = 0$. Let us now assume that $|N_{\mathbf{x}}| \geq \Gamma_X + 1$. Then, by definition, we have $|I(N_{\mathbf{x}}, \Gamma_X)| = \Gamma_X$. Let r^* be the index of the Γ_X-th largest d_i value taken into the solution, i.e., $r^* = \max\{i \,|\, i \in I(N_{\mathbf{x}}, \Gamma_X)\}$. Then we have:

$$\sum_{i \in N_{\mathbf{x}}} c_i + \sum_{i \in I(N_{\mathbf{x}}, \Gamma_X)} d_i = \sum_{i \in N_{\mathbf{x}}} c_i + \sum_{\{i \in N_{\mathbf{x}} : i \leq r^*\}} d_i - \sum_{\{i \in N_{\mathbf{x}} : i \leq r^*\}} d_{r^*} + \sum_{\{i \in N_{\mathbf{x}} : i \leq r^*\}} d_{r^*}$$

$$= \sum_{i \in N_{\mathbf{x}}} c_i + \sum_{\{i \in N_{\mathbf{x}} : i \leq r^*\}} (d_i - d_{r^*}) + \Gamma_X d_{r^*}$$

$$= \sum_{i \in I} c_i x_i + \sum_{i=1}^{r^*} (d_i - d_{r^*}) x_i + \Gamma_X d_{r^*}.$$

Note that $r^* \geq \Gamma_X$ since $|N_{\mathbf{x}}| \geq \Gamma_X + 1$. Therefore, the minimum value $ROPT_{P1}(\Gamma_X)$ will be reached for $\theta^* = d_r$, where $r \geq \Gamma_X$, and hence, $\theta^* \in [0, d_{\Gamma_X}]$, which completes the first part of the proof.

We now present the second part of the proof, where the previous result is plugged into the procedure devised in [Bertsimas and Sim(2003)], and we find the optimal values of θ by using an equivalent decomposition approach. We decompose of the real interval $[0, d_{\Gamma_X}]$ into $[0, d_n]$, $[d_n, d_{n-1}]$, ..., $[d_{\Gamma_X+1}, d_{\Gamma_X}]$. Observe that for an arbitrary $\theta \in [d_r, d_{r-1}]$ we have:

$$\sum_{i \in I} \max(d_i - \theta, 0) x_i = \sum_{i=1}^{r-1} (d_i - \theta) x_i.$$

Therefore, $ROPT_{P1}(\Gamma_X) = \min_{r \in \{\Gamma_X, \ldots, n+1\}} G^r$ where for $r \in \{\Gamma_X, \ldots, n+1\}$

$$G^r = \min \sum_{i \in I} c_i x_i + \Gamma_X \theta + \sum_{i=1}^{r-1} (d_i - \theta) x_i,$$

where $\theta \in [d_r, d_{r-1}]$ and $\mathbf{x} \in \Pi$. Since we are optimizing a linear function of θ over the interval $[d_r, d_{r-1}]$, the optimal value of G^r is obtained either by $\theta = d_r$ or by $\theta = d_{r-1}$. So, for $r \in \{\Gamma_X, \ldots, n+1\}$:

$$G^r = \min \left[\Gamma_X d_r + \min_{\mathbf{x} \in \Pi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r-1} (d_i - d_r) x_i \right), \ \Gamma_X d_{r-1} + \min_{\mathbf{x} \in \Pi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r-1} (d_i - d_{r-1}) x_i \right) \right]$$

$$= \min \left[\Gamma_X d_r + \min_{\mathbf{x} \in \Pi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r} (d_i - d_r) x_i \right), \ \Gamma_X d_{r-1} + \min_{\mathbf{x} \in \Pi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r-1} (d_i - d_{r-1}) x_i \right) \right].$$

Therefore,

$$ROPT_{P1}(\Gamma_X) = \min \left[\Gamma_X d_{\Gamma_X} + \min_{\mathbf{x} \in \Pi} \left(\sum_{i \in I} c_i x_i \right), \ldots, \Gamma_X d_r + \min_{\mathbf{x} \in \Pi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r} (d_i - d_r) x_i \right), \ldots, \right.$$

$$\left. \min_{\mathbf{x} \in \Pi} \left(\sum_{i \in I} c_i x_i + \sum_{i \in I} d_i x_i \right) \right],$$

which completes the proof. ∎

Consider now the following problem that we will refer to as (P2):

$$OPT_{P2} = \min \left\{ \sum_{i \in I} c_i x_i \mid \sum_{j \in J} b_j y_j \leq B \text{ and } (\mathbf{x}, \mathbf{y}) \in \Psi \right\}, \tag{P2}$$

where $\mathbf{y} \in \{0, 1\}^m$ are decision variables, $B \in \mathbb{R}^{\geq 0}$ is a constant, $\mathbf{b} \geq 0$, $J = \{1, 2, \ldots, m\}$, and Ψ is a generic polyhedral region.

Let us assume that $\mathbf{c}$ is known with certainty, but instead, the elements of $\mathbf{b}$ are given as uncertain intervals $[b_j, b_j + \delta_j]$, $\forall j \in J$, and that the variables are ordered so that $\delta_j \geq \delta_{j+1}$, $\forall j \in J$, and $\delta_{m+1} = 0$. Given $\Gamma_Y \in \{1, \ldots, m\}$, the robust counterpart of the nominal problem (P2), given the interval uncertainty of vector $\mathbf{b}$, is:

$$ROPT_{P2}(\Gamma_Y) = \min \left\{ \sum_{i \in I} c_i x_i \mid \sum_{j \in J} b_j y_j + \beta_Y^* (\Gamma_Y, \mathbf{y}) \leq B \text{ and } (\mathbf{x}, \mathbf{y}) \in \Psi \right\}. \tag{RP2}$$

In this case, $\beta_Y^* (\Gamma_Y, \mathbf{y})$ provides protection of feasibility in presence of a level of conservatism given by Γ_Y. This problem can be rewritten as

$$ROPT_{P2}(\Gamma_Y) = \min \sum_{i \in I} c_i x_i \tag{6}$$

$$\text{s.t} \quad \sum_{j \in J} b_j y_j + \Gamma_Y \lambda + \sum_{j \in J} k_j \leq B \tag{7}$$

$$k_j + \lambda \geq \delta_j y_j, \quad \forall j \in J \tag{8}$$

$$k_j \geq 0, \, \forall j \in J \text{ and } \lambda \geq 0 \tag{9}$$

$$(\mathbf{x}, \mathbf{y}) \in \Psi. \tag{10}$$

The following lemma extends for (RP2) the result of Theorem 3 in [Bertsimas and Sim(2003)], and adapts the result of Lemma 1.

Lemma 2 *Given $\Gamma_Y \in \{1, \ldots, m\}$, the problem* (RP2), *the robust counterpart of problem* (P2), *can be computed by solving $(m - \Gamma_Y + 2)$ nominal problems, in the following scheme:*

$$ROPT_{P2}(\Gamma_Y) = \min_{s \in \{\Gamma_Y, \ldots, m+1\}} H^s,$$

where for $s \in \{\Gamma_Y, \ldots, m+1\}$:

$$H^s = \min_{(\mathbf{x}, \mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i \mid \sum_{j \in J} b_j y_j + \sum_{j=1}^{s} (\delta_j - \delta_s) y_j + \Gamma_Y \delta_s \leq B \right).$$

Proof. The core of the proof consists of showing that for any feasible solution of (6)-(10) we have $\lambda \in [0, \delta_{\Gamma_Y}]$.

For any feasible solution of (6)-(10) holds that $k_j = \max(\delta_j y_j - \lambda, 0)$; thus, constraint (7) can be written as

$$\sum_{j \in J} b_j y_j + \Gamma_Y \lambda + \sum_{j \in J} \max(\delta_j - \lambda, 0) \, y_j \leq B. \tag{11}$$

Let $(\mathbf{x}, \mathbf{y})$ be a feasible solution for a given Γ_X and a given Γ_Y. Let $M_{\mathbf{y}}$ be a set of indices $j \in J$ such that $y_j = 1$. Let $J(M_{\mathbf{y}}, \Gamma_Y)$ be a subset of $M_{\mathbf{y}}$ associated with (at most) the Γ_Y largest values δ_j. Since $(\mathbf{x}, \mathbf{y})$ is a feasible solution, then the following holds:

$$\sum_{j \in M_{\mathbf{y}}} b_j + \sum_{j \in J(M_{\mathbf{y}}, \Gamma_Y)} \delta_j \leq B.$$

Let us assume that $|M_{\mathbf{y}}| \leq \Gamma_Y$, then we have $J(M_{\mathbf{y}}, \Gamma_Y) = M_{\mathbf{y}}$, which implies that the cost of each element corresponding to index $j \in M_{\mathbf{y}}$ will be set to its corresponding upper bound $b_j + \delta_j$, and hence constraint (11) is satisfied for $\lambda = d_{m+1} = 0$.

Let us now assume that $|M_{\mathbf{y}}| \geq \Gamma_Y + 1$. Then, by definition, we have $|J(M_{\mathbf{y}}, \Gamma_Y)| = \Gamma_Y$. Let $s^* = \max\{j \,|\, j \in J(M_{\mathbf{y}}, \Gamma_Y)\}$. So

$$
\sum_{j \in M_{\mathbf{y}}} b_j + \sum_{j \in J(M_{\mathbf{y}}, \Gamma_Y)} \delta_j = \sum_{j \in M_{\mathbf{y}}} b_j + \sum_{\{j \in M_{\mathbf{y}}: j \leq s^*\}} \delta_j - \sum_{\{j \in M_{\mathbf{y}}: j \leq s^*\}} \delta_{s^*} + \sum_{\{j \in M_{\mathbf{y}}: j \leq s^*\}} \delta_{s^*}
$$
$$
= \sum_{j \in M_{\mathbf{y}}} b_j + \sum_{\{j \in M_{\mathbf{y}}: j \leq s^*\}} (\delta_j - \delta_{s^*}) + \Gamma_Y \delta_{s^*}
$$
$$
= \sum_{j \in J} b_j y_j + \sum_{j=1}^{s^*} (\delta_j - \delta_{s^*}) y_j + \Gamma_Y \delta_{s^*} \leq B.
$$

Note that $s^* \geq \Gamma_Y$ since $|M_{\mathbf{y}}| \geq \Gamma_Y + 1$, and therefore constraint (7) will be satisfied for all $\lambda = \delta_s$ such that $s \geq \Gamma_Y$. Therefore for any feasible solution we have $\lambda \in [0, \delta_{\Gamma_Y}]$.

By following similar arguments as those presented in the decomposition approach of the poof of Lemma 1, it holds that

$$
ROPT_{P2}(\Gamma_Y) = \min \left[\min_{(\mathbf{x}, \mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i \,\Big|\, \sum_{j \in J} b_j y_j + \Gamma_Y \delta_{\Gamma_Y} \leq B \right), \dots, \right.
$$
$$
\min_{(\mathbf{x}, \mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i \,\Big|\, \sum_{j \in J} b_j y_j + \sum_{j=1}^{s} (\delta_j - \delta_s) y_j + \Gamma_Y \delta_s \leq B \right), \dots,
$$
$$
\left. \min_{(\mathbf{x}, \mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i \,\Big|\, \sum_{j \in J} b_j y_j + \sum_{j \in J} \delta_j y_j \leq B \right) \right],
$$

and the proof is completed. ∎

We now present a second extension of the algorithm proposed in [Bertsimas and Sim(2003)]. Let us consider now the following nominal problem:

$$
OPT_{P3} = \min \left\{ \sum_{i \in I} c_i x_i + \sum_{j \in J} b_j y_j \,\Big|\, (\mathbf{x}, \mathbf{y}) \in \Psi \right\}. \tag{P3}
$$

In case that the elements of both vectors $\mathbf{c}$ and $\mathbf{b}$ are given in terms of closed intervals, the corresponding robust counterpart (for a pair (Γ_X, Γ_Y)) is given by

$$
ROPT_{P3}(\Gamma_X, \Gamma_Y) = \min \sum_{i \in I} c_i x_i + \Gamma_X \theta + \sum_{i \in I} h_i + \sum_{j \in J} b_j y_j + \Gamma_Y \lambda + \sum_{j \in J} k_j \tag{12}
$$

$$
\text{s.t.} \quad (3), (4), (8), (9) \text{ and } (\mathbf{x}, \mathbf{y}) \in \Psi. \tag{13}
$$

The following result extends Lemma 1 and provides an algorithmic procedure to solve (12)-(13).

Lemma 3 *Given $\Gamma_X \in \{1, \dots, n\}$ and $\Gamma_Y \in \{1, \dots, m\}$, the robust counterpart of problem* (P3) *can be computed by solving $(n - \Gamma_X + 2)(m - \Gamma_Y + 2)$ nominal problems as follows:*

$$
ROPT_{P3}(\Gamma_X, \Gamma_Y) = \min_{\substack{r \in \{\Gamma_X, \dots, n+1\} \\ s \in \{\Gamma_Y, \dots, m+1\}}} G^{r,s},
$$

where for $r \in \{\Gamma_X, \ldots, n+1\}$ and $s \in \{\Gamma_Y, \ldots, m+1\}$:

$$G^{r,s} = \Gamma_X d_r + \Gamma_Y \delta_s + \min_{(\mathbf{x},\mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r} (d_i - d_r) x_i + \sum_{j \in J} b_j y_j + \sum_{j=1}^{s} (\delta_j - \delta_s) y_j \right).$$

Proof. Using an analogous analysis to the one in the proofs of Lemma 1 and 2, we have that for any optimal solution $(\mathbf{x}^*, \mathbf{y}^*, \theta^*, \lambda^*)$, it holds $\theta^* \in [0, d_{\Gamma_X}]$ and $\lambda^* \in [0, d_{\Gamma_Y}]$. Then, by decomposition, the optimal can be found as $ROPT_{P3}(\Gamma_X, \Gamma_Y) = \min_{\substack{r \in \{\Gamma_X, \ldots, n+1\} \\ s \in \{\Gamma_Y, \ldots, m+1\}}} G^{r,s}$ where for $r \in \{\Gamma_X, \ldots, n+1\}$ and $s \in \{\Gamma_Y, \ldots, m+1\}$

$$G^{r,s} = \min \sum_{i \in I} c_i x_i + \Gamma_X \theta + \sum_{i=1}^{r-1} (d_i - \theta) x_i + \sum_{j \in J} b_j y_j + \Gamma_Y \lambda + \sum_{i=1}^{s-1} (\delta_j - \lambda) y_j, \tag{14}$$

for which $\theta \in [d_r, d_{r-1}]$, $\lambda \in [\delta_s, \delta_{s-1}]$ and $(\mathbf{x}, \mathbf{y}) \in \Psi$. Since we are optimizing a linear function of θ over the interval $[d_r, d_{r-1}]$ and also a linear function for λ over the interval $[\delta_s, \delta_{s-1}]$, the optimal value of $G^{r,s}$ is obtained for $(\theta, \lambda) \in \{(d_r, \delta_s), (d_{r-1}, \delta_s), (d_r, \delta_{s-1}), (d_{r-1}, \delta_{s-1})\}$. So, for $r \in \{\Gamma_X, \ldots, n+1\}$ and $s \in \{\Gamma_Y, \ldots, m+1\}$:

$$
\begin{aligned}
G^{r,s} = \min \Bigg[&\Gamma_X d_r + \Gamma_Y \delta_s + \min_{(\mathbf{x},\mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r-1} (d_i - d_r) x_i + \sum_{j \in J} b_j y_j + \sum_{j=1}^{s-1} (\delta_j - \delta_s) y_j \right), \\
&\Gamma_X d_{r-1} + \Gamma_Y \delta_s + \min_{(\mathbf{x},\mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r-1} (d_i - d_{r-1}) x_i + \sum_{j \in J} b_j y_j + \sum_{j=1}^{s-1} (\delta_j - \delta_s) y_j \right), \\
&\Gamma_X d_r + \Gamma_Y \delta_{s-1} + \min_{(\mathbf{x},\mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r-1} (d_i - d_r) x_i + \sum_{j \in J} b_j y_j + \sum_{j=1}^{s-1} (\delta_j - \delta_{s-1}) y_j \right), \\
&\Gamma_X d_{r-1} + \Gamma_Y \delta_{s-1} + \min_{(\mathbf{x},\mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r-1} (d_i - d_{r-1}) x_i + \sum_{j \in J} b_j y_j + \sum_{j=1}^{s-1} (\delta_j - \delta_{s-1}) y_j \right) \Bigg] \\
= \min \Bigg[&\Gamma_X d_r + \Gamma_Y \delta_s + \min_{(\mathbf{x},\mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r} (d_i - d_r) x_i + \sum_{j \in J} b_j y_j + \sum_{j=1}^{s} (\delta_j - \delta_s) y_j \right), \\
&\Gamma_X d_{r-1} + \Gamma_Y \delta_s + \min_{(\mathbf{x},\mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r-1} (d_i - d_{r-1}) x_i + \sum_{j \in J} b_j y_j + \sum_{j=1}^{s} (\delta_j - \delta_s) y_j \right), \\
&\Gamma_X d_r + \Gamma_Y \delta_{s-1} + \min_{(\mathbf{x},\mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r} (d_i - d_r) x_i + \sum_{j \in J} b_j y_j + \sum_{j=1}^{s-1} (\delta_j - \delta_{s-1}) y_j \right), \\
&\Gamma_X d_{r-1} + \Gamma_Y \delta_{s-1} + \min_{(\mathbf{x},\mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r-1} (d_i - d_{r-1}) x_i + \sum_{j \in J} b_j y_j + \sum_{j=1}^{s-1} (\delta_j - \delta_{s-1}) y_j \right) \Bigg].
\end{aligned}
$$

Therefore,

$$ROPT_{P3}(\Gamma_X, \Gamma_Y) = \min \left[\Gamma_X d_{\Gamma_X} + \Gamma_Y \delta_{\Gamma_Y} + \min_{(\mathbf{x},\mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i + \sum_{j \in J} b_j y_j \right), \ldots, \right.$$

$$\Gamma_X d_r + \Gamma_Y \delta_s + \min_{(\mathbf{x},\mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r} (d_i - d_r) x_i + \sum_{j \in J} b_j y_j + \sum_{j=1}^{s} (\delta_j - \delta_s) y_j \right), \ldots,$$

$$\left. \min_{(\mathbf{x},\mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i + \sum_{i \in I} d_i x_i + \sum_{j \in J} b_j y_j + \sum_{j \in J} \delta_j y_j \right) \right],$$

which completes the proof. ∎

As a complementary result, one can observe that if in (P2) the cost vector $\mathbf{c}$ is also subject to interval uncertainty (along with the coefficient vector $\mathbf{b}$), the corresponding robust counterpart is given by

$$ROPT_{P4}(\Gamma_X, \Gamma_Y) = \min \sum_{i \in I} c_i x_i + \Gamma_X \theta + \sum_{i \in I} h_i \qquad (15)$$

$$\text{s.t.} \quad (3), (4), (7), (8), (9) \text{ and } (\mathbf{x}, \mathbf{y}) \in \Psi. \qquad (16)$$

Combining the results of Lemma 1 and 2, we have the following result,

Lemma 4 *Given $\Gamma_X \in \{1, \ldots, n\}$ and $\Gamma_Y \in \{1, \ldots, m\}$, the robust problem (15)-(16) can be solved by solving $(n - \Gamma_X + 2)(m - \Gamma_Y + 2)$ nominal problems as follows:*

$$ROPT_{P4}(\Gamma_X, \Gamma_Y) = \min_{\substack{r \in \{\Gamma_X, \ldots, n+1\} \\ s \in \{\Gamma_Y, \ldots, m+1\}}} H^{r,s},$$

where for $r \in \{\Gamma_X, \ldots, n+1\}$ and $s \in \{\Gamma_Y, \ldots, m+1\}$:

$$H^{r,s} = \Gamma_X d_r + \min_{(\mathbf{x},\mathbf{y}) \in \Psi} \left(\sum_{i \in I} c_i x_i + \sum_{i=1}^{r} (d_i - d_r) x_i \mid \sum_{j \in J} b_j y_j + \sum_{j=1}^{s} (\delta_j - \delta_s) y_j + \Gamma_Y \delta_s \leq B \right).$$

We omit the proof of this result as it follows from the proofs of Lemma 2 and 3.

3. General Result

In light of Lemmas 3 and 4, we now generalize the previous results considering a more general Combinatorial Optimization problem under interval uncertainty and propose a discrete algorithm to solve its robust counterpart.

Let us consider a case in which the set of binary variables is partitioned into $K + L$ subsets given by $(\mathbf{x}^1, \ldots, \mathbf{x}^K, \mathbf{y}^1, \ldots, \mathbf{y}^L)$, associated with sets of indices $(I^1, \ldots, I^K, J^1, \ldots, J^L)$. Variables $(\mathbf{x}^1, \ldots, \mathbf{x}^K)$ appear in the objective function with non-negative cost vectors $(\mathbf{c}^1, \ldots, \mathbf{c}^K)$, and $(\mathbf{y}^1, \ldots, \mathbf{y}^L)$ variables appear in L disjoint knapsack constraints with non-negative coefficients

$(\mathbf{b}^1, \ldots, \mathbf{b}^L)$ and non-negative right-hand-side bounds $(B^1, \ldots, B^L)$. Let Ψ' be a generic polyhedron containing the feasibility conditions for $(\mathbf{x}^1, \ldots, \mathbf{x}^K, \mathbf{y}^1, \ldots, \mathbf{y}^L)$. With these elements we define nominal problem (P5) as

$$OPT_{P5} = \min_{(\mathbf{x}^1,\ldots,\mathbf{y}^L) \in \Psi'} \left\{ \sum_{i \in I^1} c_i^1 x_i^1 + \ldots + \sum_{i \in I^K} c_i^K x_i^K \mid \sum_{j \in J^1} b_j^1 y_j^1 \leq B^1, \ldots, \sum_{j \in J^L} b_j^L y_j^L \leq B^L \right\}. \qquad \text{(P5)}$$

We assume now that all elements of the cost vectors $(\mathbf{c}^1, \ldots, \mathbf{c}^K)$ and all elements of the knapsack coefficients $(\mathbf{b}^1, \ldots, \mathbf{b}^L)$ are subject to interval uncertainty; the cost coefficient of variable x_i^k is taken from $[c_i^k, c_i^k + d_i^k]$, for each $i \in I^k$ and $k \in \mathcal{K} = \{1, \ldots, K\}$, and the coefficient of variable y_j^l is taken from $[b_j^l, b_j^l + \delta_j^l]$, for each $j \in J^l$ and $l \in \mathcal{L} = \{1, \ldots, L\}$. Assume that variables $(\mathbf{x}^1, \ldots, \mathbf{y}^L)$ are ordered so that $d_i^k \geq d_{i+1}^k$ and $d_{|I^k|+1}^k = 0$, for all $i \in I^k$ and $k \in \mathcal{K}$, and $\delta_j^l \geq \delta_{j+1}^l$ and $\delta_{|J^l|+1}^l = 0$, for all $j \in J^l$ and $l \in \mathcal{L}$.

To each set of cost coefficients we associate a level of conservatism $0 \leq \Gamma_X^k \leq |I^k|$, for all $k \in \mathcal{K}$, and to each knapsack constraint we associate a level of conservatism $0 \leq \Gamma_Y^l \leq |J^l|$, for all $l \in \mathcal{L}$. The following Theorem unifies the previous results.

Theorem 1 *For given $0 \leq \Gamma_X^k \leq |I^k|$, for all $k \in \mathcal{K}$, and $0 \leq \Gamma_Y^l \leq |J^l|$, for all $l \in \mathcal{L}$, the robust counterpart of* (P5), *$ROPT_{P5}(\Gamma_X^1, \ldots, \Gamma_X^K, \Gamma_Y^1, \ldots, \Gamma_Y^L)$, can be computed by solving*

$$\prod_{k \in \mathcal{K}} (|I^k| - \Gamma_X^k + 2) \prod_{l \in \mathcal{L}} (|J^l| - \Gamma_Y^l + 2)$$

problems given by

$$ROPT_{P5}(\Gamma_X^1, \ldots, \Gamma_X^K, \Gamma_Y^1, \ldots, \Gamma_Y^L) = \min_{\substack{r^1 \in \{\Gamma_X^1,\ldots,|I^1|+1\} \\ \vdots \quad \vdots \\ s^L \in \{\Gamma_Y^L,\ldots,|J^L|+1\}}} F^{(r^1,\ldots,r^K,s^1,\ldots,s^L)},$$

where for $r^1 \in \{\Gamma_X^1, \ldots, |I^1|+1\}, \ldots, r^K \in \{\Gamma_X^K, \ldots, |I^K|+1\}$ and $s^1 \in \{\Gamma_Y^1, \ldots, |J^1|+1\}, \ldots, s^L \in \{\Gamma_Y^L, \ldots, |J^L|+1\}$, we have that

$$F^{(r^1,\ldots,r^K,s^1,\ldots,s^L)} = \Gamma_x^1 d_{r^1} + \ldots + \Gamma_x^K d_{r^K} + \min_{(\mathbf{x}^1,\ldots,\mathbf{y}^K) \in \Psi'} \{\varphi^1(r^1) + \ldots + \varphi^K(r^K) \mid$$

$$\xi^1(s^1) \leq B^1, \ldots, \xi^L(s^L) \leq B^L\},$$

such that

$$\varphi^l(r^k) = \sum_{i \in I^k} c_i^k x_i^k + \sum_{i=1}^{r^k} \left(d_i^k - d_{r^k}^k \right) x_i^k, \ \forall k \in \mathcal{K},$$

and

$$\xi^l(s^l) = \sum_{j \in J^l} b_j^l y_j^l + \sum_{j=1}^{s^l} \left(\delta_j^l - \delta_{s^l}^l \right) y_j^l, \ \forall l \in \mathcal{L}.$$

Proof. The robust counterpart of (P5) can be written as

$$ROPT_{P5}(\Gamma_X^1, \ldots, \Gamma_X^K, \Gamma_Y^1, \ldots, \Gamma_Y^L) = \min \sum_{k \in \mathcal{K}} \left(\sum_{i \in I^k} c_i^k x_i^k + \Gamma_X^k \theta^k + \sum_{i \in I^k} h_i^k \right) \tag{17}$$

$$\text{s.t.} \qquad \sum_{j \in J^l} b_j^l y_j^l + \Gamma_Y^l \lambda^l + \sum_{j \in J^l} k_j^l \leq B^l, \quad l \in \mathcal{L} \tag{18}$$

$$h_i^k + \theta^k \geq d_i^k x_i^k \text{ and } \theta^k \geq 0, \quad \forall i \in I^k, \ k \in \mathcal{K} \tag{19}$$

$$k_j^l + \lambda^l \geq \delta_j^l y_j^l \text{ and } \lambda^l \geq 0, \quad \forall l \in J^l, \ l \in \mathcal{L} \tag{20}$$

$$h_i^k \geq 0, \quad \forall i \in I^k, \ k \in \mathcal{K} \tag{21}$$

$$k_j^l \geq 0, \quad \forall j \in J^l, \ l \in \mathcal{L}. \tag{22}$$

From Lemma 1 and 2, one can show by mathematical induction that any optimal solution for (17)-(22) satisfies $\theta^{k^*} \in [0, d_{\Gamma_X^k}^k]$, for each $k \in \mathcal{K}$, and $\lambda^{l^*} \in [0, \delta_{\Gamma_Y^l}^l]$, for each $l \in \mathcal{L}$. Finally, mathematical induction is applied to the previously used decomposition approach to derive the result for computing $ROPT_{P5}(\Gamma_X^1, \ldots, \Gamma_Y^L)$. ∎

As stressed in the Introduction, several Combinatorial Optimization problems are particular cases of (P5), and if interval uncertainty in their parameters is brought into play, the algorithmic procedure described by Theorem 1 could be an alternative for solving their robust counterparts.

References

[Bertsimas and Sim(2003)] D. Bertsimas and M. Sim. Robust discrete optimization and network flows. *Mathematical Programming*, B(98):49–71, 2003.